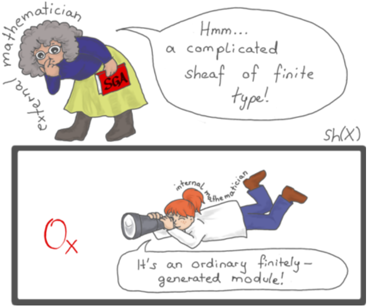

Using the internal language of toposes in algebraic geometry

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University of Augsburg

Special PSSL 99
in honor of Jirka Adámek's retirement

March 13th, 2016

Outline

Abstract

We describe how the internal language of certain toposes, the associated small and big Zariski toposes of a scheme, can be used to give simpler definitions and more conceptual proofs of the basic notions and observations in algebraic geometry.

The starting point is that, from the internal point of view, sheaves of rings and sheaves of modules look just like plain rings and plain modules. In this way, some concepts and statements of scheme theory can be reduced to concepts and statements of intuitionistic linear algebra.

Furthermore, modal operators can be used to model phrases such as "on a dense open subset it holds that" or "on an open neighbourhood of a given point it holds that". These operators define certain subtoposes; a generalization of the double-negation translation is useful in order to understand the internal universe of those subtoposes from the internal point of view of the ambient topos.

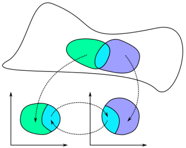
A particularly interesting task is to internalize the construction of the relative spectrum, which, given a quasicoherent sheaf of algebras on a scheme X, yields a scheme over X. From the internal point of view, this construction should simply reduce to an intuitionistically sensible variant of the ordinary construction of the spectrum of a ring, but it turns out that this expectation is too naive and that a refined approach is necessary.

What is a scheme?

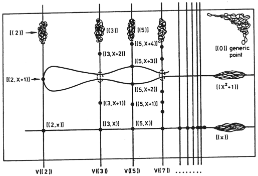
- A **manifold** is a space which is **locally isomorphic** to some open subset of some $\mathbb{R}^n$.
- A **scheme** is a space which is **locally isomorphic** to the **spectrum of some (commutative) ring**:

$$\operatorname{Spec} A := \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal}\}$$

- By **space** we mean: topological space X equipped with a local sheaf $\mathcal{O}_X$ of rings.



a manifold



Mumford's treasure map of $\operatorname{Spec} \mathbb{Z}[X]$

A *sheaf of rings* on a topological space X is a ring object in $\mathrm{Sh}(X)$, the category of set-valued sheaves on X.

A sheaf $\mathcal{O}_X$ of rings is *local* if and only if all the stalks $\mathcal{O}_{X,x}$ are local rings. Why not demand that the sets of sections $\mathcal{O}_X(U)$ are local rings? This has a geometric meaning, but can also be motivated from a logical point of view: A sheaf of rings is local if and only if, from the point of view of the internal language of $\mathrm{Sh}(X)$, it is a local ring.

Think of $\mathcal{O}_X$ as the sheaf of "number-valued functions" on X. In algebraic geometry, this structure sheaf is a crucial part of the data: Wildly different schemes can have the same underlying topological space.

What is a topos?

Formal definition

A **topos** is a category which has finite limits, is cartesian closed and has a subobject classifier.

Motto

A topos is a category which is sufficiently rich to support an **internal language**.

Examples

- **Set:** category of sets
- **Sh(X):** category of set-valued sheaves on a space X

While technically correct, the formal definition is actually misleading in a sense: A topos has lots of other vital structure, which is crucial for a rounded understanding, but is not listed in the definition (which is trimmed for minimality).

A more comprehensive definition is: A *topos* is a locally cartesian closed, finitely complete and cocomplete Heyting category which is exact, extensive and has a subobject classifier.

Check out an article by Tom Leinster for a leisurely introduction to topos theory.

What is the internal language?

Let $\mathcal{E}$ be a topos. Then we can define the meaning of

$$\mathcal{E} \models \varphi \qquad (\text{“}\varphi \text{ holds in } \mathcal{E}\text{”})$$

for formulas φ over $\mathcal{E}$ using the **Kripke–Joyal semantics**.

externally	internally to $\mathcal{E}$
object of $\mathcal{E}$	set/type
morphism in $\mathcal{E}$	map of sets
monomorphism	injective map
epimorphism	surjective map
ring object	ring
module object	module

If φ implies ψ **intuitionistically**, then $\mathcal{E} \models \varphi$ implies $\mathcal{E} \models \psi$.

- Actually, the objects of $\mathcal{E}$ feel more like *types* instead of *sets*: For instance, there is no global membership relation $\in$. Rather, for each object A of $\mathcal{E}$, there is a relation $\in_A : A \times \mathcal{P}(A) \to \Omega$, where $\mathcal{P}(A)$ is the power object of A and Ω is the object of truth values of $\mathcal{E}$ (can be understood as the power object of a terminal object).

- Compare with the embedding theorem for abelian categories: There, an explicit embedding into a category of modules is constructed. Here, we only change perspective and talk about the same objects and morphisms.

- There exists a weaker variant of the internal language which works in abelian categories. By using it, one can even pretend that the objects are abelian groups (instead of modules), and when constructing morphisms by appealing to the axiom of unique choice (which is a theorem), one doesn't even have to check linearity. The proof that this approach works uses only categorical logic.

- The translation of internal statements into external ones is facilitated by an easy mechanical procedure which one quickly grows accustomed to. ▸ See more details.

- The internal language of a sheaf topos of a T_1-space is *classical* (that is, verifies the principle of excluded middle) if and only if the space is discrete. That's a not particularly interesting special case.

- See Section 2.4 of these notes for remarks on how to appreciate intuitionistic logic.

Building a dictionary

Understand notions of algebraic geometry as notions of algebra internal to $\mathrm{Sh}(X)$.

externally	internally to $\mathrm{Sh}(X)$
sheaf of sets	set/type
morphism of sheaves	map of sets
monomorphism	injective map
epimorphism	surjective map
sheaf of rings	ring
sheaf of modules	module
sheaf of finite type	finitely generated module
finite locally free sheaf	finite free module
coherent sheaf	coherent module
tensor product of sheaves	tensor product of modules
rank function	minimal number of generators
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$

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morph	
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sheaf o	
sheaf o	
sheaf o	
finite lo	
cohere	
tensor	
rank function	minimal number of generators
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$

MISCONCEPTIONS ABOUT K_X

by Steven L. KLEIMAN

There are three common misconceptions about the sheaf K_X of meromorphic functions on a ringed space X: (1) that K_X can be defined as the sheaf associated to the presheaf of total fraction rings,

(*) $$U \mapsto \Gamma(U, O_X)_{tot},$$

see [EGA IV$_4$, 20.1.3, p. 227] and [1, (3.2), p. 137]; (2) that the stalks $K_{X,x}$ are equal to the total fraction rings $(O_{X,x})_{tot}$, see [EGA IV$_4$, 20.1.1 and 20.1.3, pp. 226-7]; and (3) that if X is a scheme and $U = \mathrm{Spec}(A)$ is

See the notes for more dictionary entries.

The simple definition of $\mathcal{K}_X$ allows to give an internal account of the basics of the theory of Cartier divisors, for instance giving an easy description of the line bundle associated to a Cartier divisor.

Using the dictionary

Let X be a scheme. Employ its **small Zariski topos**: $\mathrm{Sh}(X)$.

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If M' and M'' are finitely generated, so is M.

$$\Downarrow$$

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are of finite type, so is $\mathcal{F}$.

Using the dictionary

Any finitely generated vector space does *not not* possess a basis.

$$\Downarrow$$

Any sheaf of modules of finite type on a reduced scheme is locally free *on a dense open subset*.

Ravi Vakil: "Important hard exercise" (13.7.K).

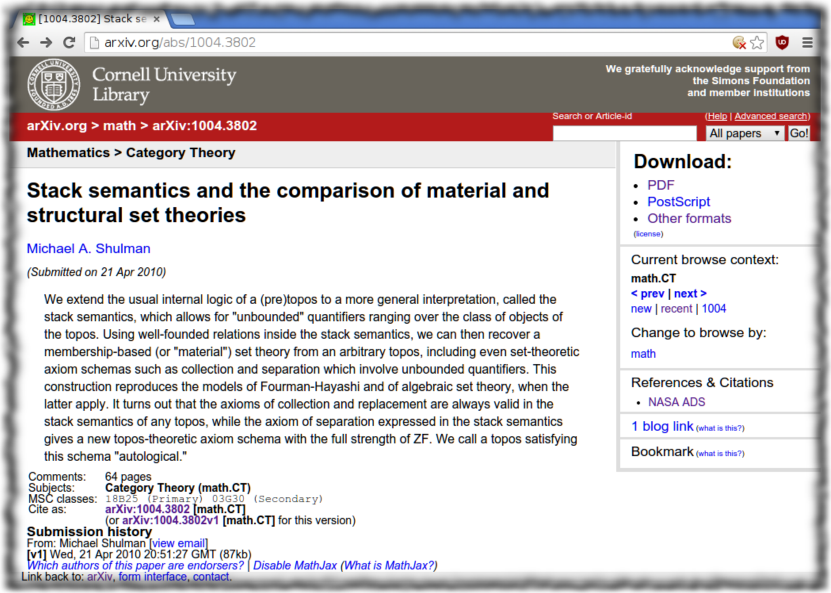
The objective

Understand notions and statements of **algebraic geometry** as notions and statements of (intuitionistic) **commutative algebra** internal to suitable **toposes**.

Further examples:

- Characterizing quasicoherence internally
- Understanding spreading of properties in a logical way
- Constructing the relative spectrum internally

Praise for Mike Shulman



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Mathematics > Category Theory

Stack semantics and the comparison of material and structural set theories

Michael A. Shulman

(Submitted on 21 Apr 2010)

We extend the usual internal logic of a (pre)topos to a more general interpretation, called the stack semantics, which allows for "unbounded" quantifiers ranging over the class of objects of the topos. Using well-founded relations inside the stack semantics, we can then recover a membership-based (or "material") set theory from an arbitrary topos, including even set-theoretic axiom schemas such as collection and separation which involve unbounded quantifiers. This construction reproduces the models of Fourman-Hayashi and of algebraic set theory, when the latter apply. It turns out that the axioms of collection and replacement are always valid in the stack semantics of any topos, while the axiom of separation expressed in the stack semantics gives a new topos-theoretic axiom schema with the full strength of ZF. We call a topos satisfying this schema "autological."

Comments: 64 pages
Subjects: **Category Theory (math.CT)**
MSC classes: 18B25 (Primary) 03G30 (Secondary)
Cite as: arXiv:1004.3802 [math.CT]
(or arXiv:1004.3802v1 [math.CT] for this version)

Submission history
From: Michael Shulman [view email]
[v1] Wed, 21 Apr 2010 20:51:27 GMT (87kb)

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The internal language of a topos supports

- first-order logic,
- higher-order logic (for instance quantification over subsets),
- dependent types, and
- unbounded quantification.

The first three items are standard. The fourth is due to Mike Shulman. Combined, it's possible to interpret "essentially all of constructive mathematics" internal to a topos.

Restrictions persist for operations with a "set-theoretical flavor" like building an infinite union of iterated powersets, for example $\bigcup_{n\in\mathbb{N}} P^n(\mathbb{N})$.

A curious property

Let X be a scheme. Internally to $\mathrm{Sh}(X)$,

any non-invertible element of $\mathcal{O}_X$ is nilpotent.

ON THE SPECTRUM OF A RINGED TOPOS 209

For completeness, two further remarks should be added to this treatment of the spectrum. One is that in **E** the canonical map $A \to \Gamma_*(LA)$ is an isomorphism—i.e., the representation of A in the ring of "global sections" of LA is complete. The second, due to Mulvey in the case $\mathbf{E} = \mathbf{S}$, is that in $\mathrm{Spec}(\mathbf{E}, A)$ the formula

$$\neg(x \in U(LA)) \Rightarrow \exists n(x^n = 0)$$

is valid. This is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas. In any case, calculations such as these are easier from the point of view of the Heyting algebra of radical ideals of A, and hence will be omitted here.

Miles Tierney. On the spectrum of a ringed topos. 1976.

Quasicoherence

Let X be a scheme. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module.

Then $\mathcal{E}$ is quasicoherent if and only if, internally to $\mathrm{Sh}(X)$,

$$\mathcal{E}[f^{-1}] \text{ is a } \Diamond_f\text{-sheaf for any } f : \mathcal{O}_X,$$
$$\text{where } \Diamond_f \varphi :\equiv (f \text{ invertible} \Rightarrow \varphi).$$

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$$\text{where } \Diamond_f \varphi :\equiv (f \text{ invertible} \Rightarrow \varphi).$$

In particular: If $\mathcal{E}$ is quasicoherent, then internally

$$(f \text{ invertible} \Rightarrow s = 0) \Longrightarrow \bigvee_{n \geq 0} f^n s = 0$$

for any $f : \mathcal{O}_X$ and $s : \mathcal{E}$.

The sheaf condition and the sheafification functor can be described purely internally. An object M is *separated* with respect to $\Diamond$ if and only if, from the internal point of view,

$$\forall x, y : M.\ \Diamond(x = y) \Rightarrow x = y.$$

It is a *sheaf* with respect to $\Diamond$, if furthermore

$$\forall K \subseteq M.\ \Diamond(\exists x : M.\ K = \{x\}) \Longrightarrow \exists x : M.\ \Diamond(x \in K).$$

The second condition displayed on the previous slide is equivalent to the separatedness condition. In the special case $\mathcal{E} = \mathcal{O}_X$, $s = 1$ it reduces to Mulvey's "somewhat obscure formula". We now understand this condition in its proper context.

The $\Diamond$-translation

Let $\mathcal{E}_\Diamond \hookrightarrow \mathcal{E}$ be a subtopos given by a local operator $\Diamond$.
Then

$$\mathcal{E}_\Diamond \models \varphi \qquad \text{iff} \qquad \mathcal{E} \models \varphi^\Diamond,$$

$\Diamond : \Omega_\mathcal{E} \to \Omega_\mathcal{E}$

where the translation $\varphi \mapsto \varphi^\Diamond$ is given by:

$$\begin{aligned}
(s = t)^\Diamond &:\equiv \Diamond(s = t) \\
(\varphi \wedge \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \wedge \psi^\Diamond) \\
(\varphi \vee \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \vee \psi^\Diamond) \\
(\varphi \Rightarrow \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \Rightarrow \psi^\Diamond) \\
(\forall x : X.\, \varphi(x))^\Diamond &:\equiv \Diamond(\forall x : X.\, \varphi^\Diamond(x)) \\
(\exists x : X.\, \varphi(x))^\Diamond &:\equiv \Diamond(\exists x : X.\, \varphi^\Diamond(x))
\end{aligned}$$

The $\Diamond$-translation

Let $\mathcal{E}_\Diamond \hookrightarrow \mathcal{E}$ be a subtopos given by a local operator $\Diamond$.
Then

$$\mathcal{E}_\Diamond \models \varphi \qquad \text{iff} \qquad \mathcal{E} \models \varphi^\Diamond.$$

$\Diamond : \Omega_\mathcal{E} \to \Omega_\mathcal{E}$

Let X be a scheme. Depending on $\Diamond$, $\mathrm{Sh}(X) \models \Diamond\varphi$ means that φ holds on …

- … a dense open subset.
- … a schematically dense open subset.
- … a given open subset U.
- … an open subset containing a given closed subset A.
- … an open neighbourhood of a given point $x \in X$.

Can tackle the question "$\varphi^\Diamond \overset{?}{\Rightarrow} \Diamond\varphi$" logically.

The $\Diamond$-translation is a generalization of the *double negation translation*, which is well-known in logic. The double negation translation has the following curious property: A formula φ admits a classical proof if and only if the translated formula $\varphi^{\neg\neg}$ admits an intuitionistic proof.

The $\Diamond$-translation has been studied before (see for instance Aczel: *The Russell–Prawitz modality*, and Escardó, Oliva: *The Peirce translation and the double negation shift*), but to the best of my knowledge, this application – expressing the internal language of subtoposes in the internal language of the ambient topos – is new.

For ease of exposition, assume that X is irreducible with generic point ξ. Let $\Diamond :\equiv \neg\neg$.

Then $\mathrm{Sh}(X) \models \Diamond\varphi$ means that φ holds on a dense open subset of X, while $\mathrm{Sh}(X) \models \varphi^{\Diamond}$ means that φ holds at the generic point (taking stalks of all involved sheaves).

The question "does $\varphi^{\Diamond}$ imply $\Diamond\varphi$?" therefore means: Does φ spread from the generic point to a dense open subset?

For the special case of the double negation translation, a general answer to this purely logical question has long been known: This holds if φ is a *geometric formula* (doesn't contain $\Rightarrow$ and $\forall$).

Let $\mathcal{F}$ be a sheaf of modules on a locally ringed space X. Assume that the stalk $\mathcal{F}_x$ at some point $x \in X$ vanishes. Then in general it does *not* follow that $\mathcal{F}$ vanishes on some open neighbourhood of x.

This can be understood in logical terms: The statement that $\mathcal{F}$ vanishes,

$$\forall s : \mathcal{F}.\ s = 0,$$

is not a geometric formula.

However, if $\mathcal{F}$ is additionally supposed to be of finite type, then it *does* follow that $\mathcal{F}$ vanishes on an open neighbourhood. This too can be understood in logical terms: If $\mathcal{F}$ is of finite type, then internally there are generators $s_1, \ldots, s_n$ of $\mathcal{F}$. Thus the vanishing of $\mathcal{F}$ can be reformulated as

$$s_1 = 0 \wedge \cdots \wedge s_n = 0,$$

and this condition is manifestly geometric.

The absolute spectrum

Let A be a commutative ring (in Set).

Is there a **free local ring** $A \to A'$ over A?

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] & & \overset{\text{local}}{R} \\
& \underset{\text{local}}{A'} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

No, if we restrict to Set.

Yes, if we allow a change of topos: Then $A \to \mathcal{O}_{\operatorname{Spec} A}$ is the universal localization.

Details on this point of view can be found in one of Peter Arndt's very nice answers on MathOverflow:

http://mathoverflow.net/a/14334/31233

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\operatorname{Spec} A := \text{topological space of the prime ideals of } A$$

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Define the frame of opens of $\operatorname{Spec} A$ to be the frame of radical ideals in A.

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This gives an internal description of Monique Hakim's spectrum functor from ringed toposes to locally ringed toposes.

Monique Hakim constructed in her thesis a very general spectrum functor, taking a ringed topos to a locally ringed one, using explicit calculations with sites.

Using the internal language allows to reduce these calculations to a minimum. One constructs the spectrum as the sheaf topos over an internal locale and then uses the general theorem that toposes over the base $\mathcal{E}$ are the same as toposes internal to $\mathcal{E}$.

As a byproduct one obtains that Hakim's spectrum is *localic* over the base.

The relative spectrum

Let X be a scheme and $\mathcal{O}_X \xrightarrow{\varphi} \mathcal{A}$ be a quasicoherent algebra. Can we describe $\boldsymbol{\underline{\mathrm{Spec}}_X\, \mathcal{A}}$, a scheme over X, internally?

Desired universal property:

$$\mathrm{Hom}_{\mathrm{Sch}/X}(T, \underline{\mathrm{Spec}}_X\, \mathcal{A}) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_* \mathcal{O}_T)$$

for all X-schemes $T \xrightarrow{\mu} X$.

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Solution: Define internally the frame of $\underline{\mathrm{Spec}}_X\,\mathcal{A}$ to be the frame of those radical ideals $I \subseteq \mathcal{A}$ such that

$$\forall f : \mathcal{O}_X.\, \forall s : \mathcal{A}.\, (f \text{ invertible in } \mathcal{O}_X \Rightarrow s \in I) \Longrightarrow fs \in I.$$

The relative spectrum

Let X be a scheme and $\mathcal{O}_X \xrightarrow{\varphi} \mathcal{A}$ be a quasicoherent algebra. Can we describe **$\underline{\mathrm{Spec}}_X \mathcal{A}$**, a scheme over X, internally?

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$$\forall f : \mathcal{O}_X.\, \forall s : \mathcal{A}.\, (f \text{ invertible in } \mathcal{O}_X \Rightarrow s \in I) \Longrightarrow fs \in I.$$

Its **points** are those prime filters G of $\mathcal{A}$ such that

$$\forall f : \mathcal{O}_X.\, \varphi(f) \in G \Longrightarrow f \text{ invertible in } \mathcal{O}_X.$$

The stated condition on I is, under the assumption that $\mathcal{A}$ is quasicoherent, equivalent to the condition that I is quasicoherent (as an $\mathcal{O}_X$-module).

The relative spectrum is thus constructed as a certain sublocale of the absolute one. The two constructions coincide if and only if the dimension of the base scheme is ≤ 0.

If X is not a scheme or $\mathcal{A}$ is not quasicoherent, the construction still gives rise to a locally ringed locale over X which satisfies the universal property

$$\mathrm{Hom}_{\mathrm{LRL}/X}(T, \underline{\mathrm{Spec}}_X \mathcal{A}) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_* \mathcal{O}_T)$$

for all locally ringed locales $T \xrightarrow{\mu} X$ over X.

The relative spectrum, reformulated

Let $B \to A$ be an algebra in topos.

Is there a **free local and local-over-B ring** $A \to A'$ over A?

$$\begin{tikzcd}
B \arrow[r] \arrow[rrr, bend left, "\text{local}"] \arrow[drr, bend right, "\text{local}"'] & A \arrow[rr] \arrow[dr] & & \overset{\text{local}}{R} \\
& & \underset{\text{local}}{A'} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

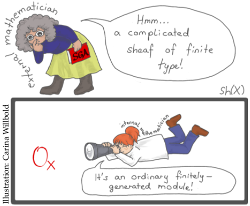
Form limits in the category of **locally ringed locales** by **relocalizing** the corresponding limit in ringed locales.

One might wonder whether the absolute spectrum or the relative one is “more fundamental”. The absolute spectrum can be expressed using the relative one, since

$$\operatorname{Spec} A = \underline{\operatorname{Spec}}_{\operatorname{Spec} \mathbb{Z}} A^{\sim},$$

but the other way is not in general possible: The absolute spectrum is always (quasi-)compact, while the relative one is not in general.

Understand notions and statements of algebraic geometry as notions and statements of algebra internal to appropriate toposes.



- Simplify proofs and gain conceptual understanding.
- Understand relative geometry as absolute geometry.
- Develop a synthetic account of scheme theory.
- Contribute to constructive algebra.

http://tiny.cc/topos-notes

spreading of properties, general transfer principles, applications to constructive algebra, quasicoherence, internal Cartier divisors, pullback along immersions = internal sheafification, scheme dimension = internal Krull dimension of $\mathcal{O}_X$, dense = not not, modal operators, relative spectrum, other toposes, étale topology, group schemes = groups, ...

You should totally look up:

The Adventures of Sheafification Man

More on the internal language

More generally, for an object U of a topos $\mathcal{E}$, we define the meaning of

$$U \models \varphi \qquad (\varphi \text{ holds on } U).$$

Writing "$\mathcal{E} \models \varphi$" is then an abbreviation for "$1 \models \varphi$", where "1" denotes the terminal object of $\mathcal{E}$.

In addition to soundness with respect to intuitionistic logic, the internal language has the following two important properties:

- **Monotonicity:** If $p : V \to U$ is an arbitrary morphism and $U \models \varphi$, then also $V \models \varphi$.
- **Locality:** If $p : V \to U$ is an epimorphism and $V \models \varphi$, then also $U \models \varphi$.

The rules of the Kripke–Joyal semantics

In the special case that $\mathcal{E} = \mathrm{Sh}(X)$ is the topos of sheaves on a topological space (or locale) X, the rules of the Kripke–Joyal semantics look as follows. We tersely write “$U \models \varphi$” instead of “$\mathrm{Hom}(\_, U) \models \varphi$ for open subsets $U \subseteq X$.

$$
\begin{aligned}
&U \models f = g : \mathcal{F} && :\Longleftrightarrow && f|_U = g|_U \in \mathcal{F}(U) \\
&U \models \varphi \wedge \psi && :\Longleftrightarrow && U \models \varphi \text{ and } U \models \psi \\
&U \models \varphi \vee \psi && :\Longleftrightarrow && \cancel{U \models \varphi \text{ or } U \models \psi} \\
& && && \text{there exists a covering } U = \textstyle\bigcup_i U_i \text{ s. th. for all } i\text{:} \\
& && && \qquad U_i \models \varphi \text{ or } U_i \models \psi \\
&U \models \varphi \Rightarrow \psi && :\Longleftrightarrow && \text{for all open } V \subseteq U\text{: } V \models \varphi \text{ implies } V \models \psi \\
&U \models \forall f : \mathcal{F}.\, \varphi(f) && :\Longleftrightarrow && \text{for all sections } f \in \mathcal{F}(V), V \subseteq U\text{: } V \models \varphi(f) \\
&U \models \exists f : \mathcal{F}.\, \varphi(f) && :\Longleftrightarrow && \text{there exists a covering } U = \textstyle\bigcup_i U_i \text{ s. th. for all } i\text{:} \\
& && && \qquad \text{there exists } f_i \in \mathcal{F}(U_i) \text{ s. th. } U_i \models \varphi(f_i)
\end{aligned}
$$

- The rules are not all arbitrary: Rather, they are very finely concerted to make the crucial properties about the internal language (monotonicity, locality, soundness with respect to intuitinistic logic) true.

- If $\mathcal{F}$ is an object of $\mathrm{Sh}(X)$, we write "$f : \mathcal{F}$" instead of "$f \in \mathcal{F}$" to remind us that $\mathcal{F}$ is not really (externally) a set consisting of elements, but that we only pretend this by using the internal language.

- There are two further rules concerning the constants $\top$ and $\perp$ (truth resp. falsehood):

$$\begin{aligned} U \models \top \quad &:\Longleftrightarrow \quad U = U \text{ (always fulfilled)} \\ U \models \perp \quad &:\Longleftrightarrow \quad U = \emptyset \end{aligned}$$

- Negation is defined as

$$\neg\varphi :\equiv (\varphi \Rightarrow \perp).$$

The alternate definition "$U \models \varphi \vee \psi :\Leftrightarrow U \models \varphi$ or $U \models \psi$" would not be local.

Translating internal statements I

Let X be a topological space (or locale) and let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then:

$$\mathrm{Sh}(X) \models \ulcorner \alpha \text{ is injective} \urcorner$$

$$\iff \mathrm{Sh}(X) \models \forall s : \mathcal{F}.\, \forall t : \mathcal{F}.\, \alpha(s) = \alpha(t) \Rightarrow s = t$$

$\iff$ for all open $U \subseteq X$, sections $s \in \mathcal{F}(U)$:
for all open $V \subseteq U$, sections $t \in \mathcal{F}(V)$:
for all open $W \subseteq V$:
$\alpha_W(s|_W) = \alpha_W(t|_W)$ implies $s|_W = t|_W$

$\iff$ for all open $U \subseteq X$, sections $s, t \in \mathcal{F}(U)$:
$\alpha_U(s|_U) = \alpha_U(t|_U)$ implies $s|_U = t|_U$

$\iff$ α is a monomorphism of sheaves

Translating internal statements II

Let X be a topological space (or locale) and let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then:

$$
\begin{aligned}
& \mathrm{Sh}(X) \models \ulcorner \alpha \text{ is surjective} \urcorner \\
\Longleftrightarrow\ & \mathrm{Sh}(X) \models \forall t\,{:}\,\mathcal{G}.\ \exists s\,{:}\,\mathcal{F}.\ \alpha(s) = t \\
\Longleftrightarrow\ & \text{for all open } U \subseteq X, \text{ sections } t \in \mathcal{G}(U): \\
& \qquad \text{there exists an open covering } U = \textstyle\bigcup_i U_i \text{ and} \\
& \qquad \text{sections } s_i \in \mathcal{F}(U_i) \text{ such that:} \\
& \qquad\qquad \alpha|_{U_i}(s_i) = t|_{U_i} \\
\Longleftrightarrow\ & \alpha \text{ is an epimorphism of sheaves}
\end{aligned}
$$

Translating internal statements III

Let X be a topological space (or locale) and let $s, t \in \mathcal{F}(X)$ be global sections of a sheaf $\mathcal{F}$ on X. Then:

$$\begin{aligned}
&\mathrm{Sh}(X) \models \neg\neg(s = t) \\
\iff &\mathrm{Sh}(X) \models ((s = t) \Rightarrow \bot) \Rightarrow \bot \\
\iff &\text{for all open } U \subseteq X \text{ such that} \\
&\qquad \text{for all open } V \subseteq U \text{ such that} \\
&\qquad\qquad s|_V = t|_V, \\
&\qquad \text{it holds that } V = \emptyset, \\
&\text{it holds that } U = \emptyset \\
\iff &\text{there exists a dense open set } W \subseteq X \text{ such that } s|_W = t|_W
\end{aligned}$$

Spreading from points to neighbourhoods

All of the following lemmas have a short, sometimes trivial proof. Let $\mathcal{F}$ be a sheaf of finite type on a ringed space X. Let $x \in X$. Let $A \subseteq X$ be a closed subset. Then:

1. $\mathcal{F}_x = 0$ iff $\mathcal{F}|_U = 0$ for some open neighbourhood of x.
2. $\mathcal{F}|_A = 0$ iff $\mathcal{F}|_U = 0$ for some open set containing A.
3. $\mathcal{F}_x$ can be generated by n elements iff this is true on some open neighbourhood of x.
4. $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ if $\mathcal{F}$ is of finite presentation around x.
5. $\mathcal{F}$ is torsion iff $\mathcal{F}_\xi$ vanishes (assume X integral and $\mathcal{F}$ quasicoherent).
6. $\mathcal{F}$ is torsion iff $\mathcal{F}|_{\mathrm{Ass}(\mathcal{O}_X)}$ vanishes (assume X locally Noetherian and $\mathcal{F}$ quasicoherent).

Statements 1 and 2 follow from *one* proof in the internal language, applied to two different modal operators.

Similarly with statements 5 and 6.

The smallest dense sublocale

Let X be a reduced scheme satisfying a technical condition. Let $i : X_{\neg\neg} \to X$ be the inclusion of the smallest dense sublocale of X.

Then $i_* i^{-1} \mathcal{O}_X \cong \mathcal{K}_X$.

- This is a highbrow way of saying "rational functions are regular functions which are defined on a dense open subset".
- Another reformulation is that $\mathcal{K}_X$ is the sheafification of $\mathcal{O}_X$ with respect to the $\neg\neg$-modality.
- There is a generalization to nonreduced schemes.

Transfer principles

Let M be an A-module. How do M and the sheaf $M^\sim$ on $\operatorname{Spec} A$ relate?

Observe that $M^\sim \cong \underline{M}[\mathcal{F}^{-1}]$ is the localization of M at the **generic prime filter** and that M shares all first-order properties with the constant sheaf of modules $\underline{M}$. Therefore:

$M^\sim$ inherits all those properties of M which are **stable under localization**.

Examples: finitely generated, free, flat, …

A converse holds as well, suitably formulated.

Applications in algebra

Let A be a commutative ring. The internal language of $\mathrm{Sh}(\operatorname{Spec} A)$ allows you to say "without loss of generality, we may assume that A is local", even constructively.

The kernel of any matrix over a principial ideal domain is finitely generated.

$$\Downarrow$$

The kernel of any matrix over a Prüfer domain is finitely generated.

Hilbert's program in algebra

There is a way to combine some of the powerful tools of classical ring theory with the advantages that constructive reasoning provides, for instance exhibiting explicit witnesses. Namely we can devise a language in which we can usefully talk about prime ideals, but which substitutes non-constructive arguments by constructive arguments "behind the scenes". The key idea is to substitute the phrase "for all prime ideals" (or equivalently "for all prime filters") by "for the generic prime filter".

More specifically, simply interpret a given proof using prime filters in $\mathrm{Sh}(\operatorname{Spec} A)$ and let it refer to $\mathcal{F} \hookrightarrow \underline{A}$.

Statement	constructive substitution	meaning
$x \in \mathfrak{p}$ for all $\mathfrak{p}$.	$x \notin \mathcal{F}$.	x is nilpotent.
$x \in \mathfrak{p}$ for all $\mathfrak{p}$ such that $y \in \mathfrak{p}$.	$x \in \mathcal{F} \Rightarrow y \in \mathcal{F}$.	$x \in \sqrt{(y)}$.
x is regular in all stalks $A_\mathfrak{p}$.	x is regular in $\underline{A}[\mathcal{F}^{-1}]$.	x is regular in A.
The stalks $A_\mathfrak{p}$ are reduced.	$\underline{A}[\mathcal{F}^{-1}]$ is reduced.	A is reduced.
The stalks $M_\mathfrak{p}$ vanish.	$\underline{M}[\mathcal{F}^{-1}] = 0$.	$M = 0$.
The stalks $M_\mathfrak{p}$ are flat over $A_\mathfrak{p}$.	$\underline{M}[\mathcal{F}^{-1}]$ is flat over $\underline{A}[\mathcal{F}^{-1}]$.	M is flat over A.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are injective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is injective.	$M \to N$ is injective.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are surjective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is surjective.	$M \to N$ is surjective.

This is related (in a few cases equivalent) to the *dynamical methods in algebra* explored by Coquand, Coste, Lombardi, Roy, and others. Their approach is more versatile.

The big Zariski topos

Let X be a scheme. The **big Zariski topos** is the topos of sheaves on Sch/X with respect to the Zariski topology. From its point of view, …

- … X-schemes look just like sets,
- … $\mathbb{P}^n_X$ is given by the naive expression

$$\{(x_0,\ldots,x_n) \mid x_1 \neq 0 \vee \cdots \vee x_n \neq 0\}/(\text{rescaling}),$$

- … the cotangent "bundle" of an X-scheme T is

$$\text{the set of maps } \Delta \to \underline{T},$$

  where $\Delta = \{\varepsilon \in \underline{\mathbb{A}}^1_X \mid \varepsilon^2 = 0\}$.
- … affinity is a "double dual condition", and
- … the étale topology is the coarsest topology $\Diamond$ s. th.

$$\forall f : \underline{\mathbb{A}}^1_X[T].\ f \text{ is monic separable} \Rightarrow \Diamond(\exists t : \underline{\mathbb{A}}^1. f(t) = 0).$$

- The functor of points of $\mathbb{A}^1_X$, that is

$$\underline{\mathbb{A}}^1_X : (T/X) \longmapsto \mathcal{O}_T(T),$$

looks like a local ring and indeed like a field from the internal point of view, in the sense that

$$\forall f : \underline{\mathbb{A}}^1_X.\, \neg(f = 0) \Rightarrow f \text{ invertible}.$$

- Let $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra. Let $\mathcal{E}$ be the induced $\underline{\mathbb{A}}^1_X$-algebra given by $\mathcal{E}(T \xrightarrow{\mu} X) := (\mu^*\mathcal{A})(T)$. Then the internal Hom set $[\mathcal{E}, \underline{\mathbb{A}}^1_X]_{\underline{\mathbb{A}}^1_X}$ of $\underline{\mathbb{A}}^1_X$-algebra morphisms is the functor of points of $\underline{\mathrm{Spec}}_X(\mathcal{A})$.

- Let $\mu : T \to X$ be quasicompact and quasiseparated. Then μ is affine iff, from the internal point of view, the map

$$\underline{T} \longrightarrow [[\underline{T}, \underline{\mathbb{A}}^1_X]^\flat, \underline{\mathbb{A}}^1_X]_{\underline{\mathbb{A}}^1_X},\ x \longmapsto \_(x)$$

into the "double dual" is bijective.

- Describing the functor of points of the projective space was suggested by Zhen Lin Low.
- The statement on the étale topology follows from Gavin Wraith's article *Generic Galois theory of local rings*.