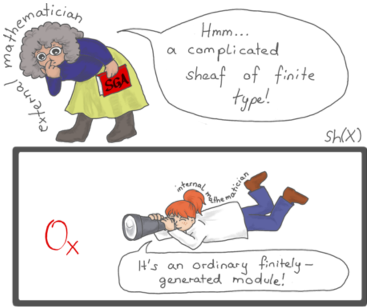

Synthetic algebraic geometry

a case study in applied topos theory

&

the phenomenon of nongeometric sequents

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Approaches to algebraic geometry

Usual approach to algebraic geometry: **layer schemes above ordinary set theory** using either

- locally ringed spaces

  set of prime ideals of $\mathbb{Z}[X, Y, Z]/(X^n + Y^n - Z^n) +$ Zariski topology + structure sheaf

- or Grothendieck's functor-of-points account, where a scheme is a functor Ring $\to$ Set.

$$A \longmapsto \{(x, y, z) \in A^3 \mid x^n + y^n - z^n = 0\}$$

Synthetic approach: model schemes **directly as sets** in a certain nonclassical set theory.

$$\{(x, y, z) : (\underline{\mathbb{A}}^1)^3 \mid x^n + y^n - z^n = 0\}$$

Toposes as mathematical universes

A **topos** is a category which has finite limits, is cartesian closed and has a subobject classifier, for instance

- **Set**, the category of sets;
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- **Eff**, the effective topos (roughly: a category of data types).

Any topos supports an **internal language**, which is sound with respect to **intuitionistic reasoning**.

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no $\varphi \vee \neg\varphi$, no $\neg\neg\varphi \Rightarrow \varphi$, no axiom of choice

Curious universes

- Eff $\models$ "There are infinitely many prime numbers." ✓
  External meaning: There is a **Turing machine** producing arbitrarily many prime numbers.

- Eff $\models$ "Any Turing machine halts or doesn't halt." ✗
  External meaning: There is a **halting oracle** which determines whether any given machine halts or doesn't halt.

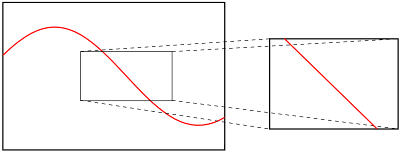
- $\mathrm{Sh}(X) \models$ "Any cont. function with opposite signs has a zero." ✗
  External meaning: Zeros can locally be picked **continuously** in continuous families of continuous functions.

(see video for counterexample)

Synthetic differential geometry

The axiom of microaffinity

Let $\Delta = \{\varepsilon \in \mathbb{R} \mid \varepsilon^2 = 0\}$. For any function $f : \Delta \to \mathbb{R}$, there are unique numbers $a, b \in \mathbb{R}$ such that $f(\varepsilon) = a + b\varepsilon$ for all $\varepsilon \in \Delta$.



- The **derivative** of f as above at zero is b.
- Manifolds are **just sets**.
- A **tangent vector** to M is a map $\Delta \to M$.

Toposes provide models for this theory.

The big Zariski topos

Let S be a fixed base scheme.

Definition

The **big Zariski topos** $\mathrm{Zar}(S)$ is the category $\mathrm{Sh}(\mathrm{Sch}/S)$. It consists of functors $(\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$ satisfying the gluing condition that

$$F(T) \to \prod_i F(U_i) \rightrightarrows \prod_{j,k} F(U_j \cap U_k)$$

is a limit diagram for any scheme $T = \bigcup_i U_i$ over S.

- For an S-scheme X, its functor of points $\underline{X} = \mathrm{Hom}_S(\cdot, X)$ is an object of $\mathrm{Zar}(S)$. It feels like **the set of points** of X.
- In particular, there is the ring object $\underline{\mathbb{A}}^1$ with $\underline{\mathbb{A}}^1(T) = \mathcal{O}_T(T)$.
- $\mathrm{Zar}(S)$ classifies local $\mathcal{O}_S$-algebras which are local over $\mathcal{O}_S$.

Synthetic constructions

$\mathbb{A}^n = (\underline{\mathbb{A}}^1)^n = \underline{\mathbb{A}}^1 \times \cdots \times \underline{\mathbb{A}}^1$

$\mathbb{P}^n = \{(x_0, \ldots, x_n) : (\underline{\mathbb{A}}^1)^{n+1} \mid x_0 \neq 0 \vee \cdots \vee x_n \neq 0\} / (\underline{\mathbb{A}}^1)^\times$
$\cong$ set of one-dimensional subspaces of $(\underline{\mathbb{A}}^1)^{n+1}$
(with $\mathcal{O}(-1) = (\ell)_{\ell : \mathbb{P}^n}$, $\mathcal{O}(1) = (\ell^\vee)_{\ell : \mathbb{P}^n}$)

$\mathbf{Spec}(R) = \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1)}(R, \underline{\mathbb{A}}^1)$ = set of $\underline{\mathbb{A}}^1$-valued points of R

$TX = X^\Delta$, where $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1 \mid \varepsilon^2 = 0\}$

A subset $U \subseteq X$ is **qc-open** if and only if for any $x : X$ there exist $f_1, \ldots, f_n : \underline{\mathbb{A}}^1$ such that $x \in U \iff \exists i.\, f_i \neq 0$.

A **synthetic affine scheme** is a set which is in bijection with $\mathrm{Spec}(R)$ for some synthetically quasicoherent $\underline{\mathbb{A}}^1$-algebra R.

A **synthetic scheme** is a set which can be covered by finitely many qc-open synthetic affine schemes U_i such that the intersections $U_i \cap U_j$ can be covered by finitely many qc-open synthetic affine schemes.

Properties of the affine line

- $\underline{\mathbb{A}}^1$ is a local ring:

$$1 \neq 0 \qquad\qquad x + y \text{ inv.} \Longrightarrow x \text{ inv.} \vee y \text{ inv.}$$

- $\underline{\mathbb{A}}^1$ is a field:

$$\neg(x = 0) \Longleftrightarrow x \text{ invertible} \qquad \text{[Kock 1976]}$$
$$\neg(x \text{ invertible}) \Longleftrightarrow x \text{ nilpotent}$$

- $\underline{\mathbb{A}}^1$ satisfies the axiom of microaffinity: Any map $f : \Delta \to \underline{\mathbb{A}}^1$ is of the form $f(\varepsilon) = a + b\varepsilon$ for unique values $a, b : \underline{\mathbb{A}}^1$, where $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1 \mid \varepsilon^2 = 0\}$.
- Any function $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is a polynomial.
- $\underline{\mathbb{A}}^1$ is anonymously algebraically closed: Any monic polynomial does *not not* have a zero.
- $\underline{\mathbb{A}}^1$ is of unbounded Krull dimension.

Synthetic quasicoherence

Recall $\mathrm{Spec}(R) = \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1)}(R, \underline{\mathbb{A}}^1)$ and consider the statement

$$\text{“the canonical map}\quad \begin{array}{rcl} R & \longrightarrow & (\underline{\mathbb{A}}^1)^{\mathrm{Spec}(R)} \\ f & \longmapsto & (\alpha \mapsto \alpha(f)) \end{array} \quad \text{is bijective”.}$$

- True for $R = \underline{\mathbb{A}}^1[X]/(X^2)$ (microaffinity).
- True for $R = \underline{\mathbb{A}}^1[X]$ (every function is a polynomial).
- True for **any** finitely presented $\underline{\mathbb{A}}^1$-algebra R.

Any known property of $\underline{\mathbb{A}}^1$ follows from this **synthetic quasicoherence**.

Example. Let $x : \underline{\mathbb{A}}^1$ such that $x \neq 0$. Set $R = \underline{\mathbb{A}}^1/(x)$.
Then $\mathrm{Spec}(R) = \emptyset$. Thus $(\underline{\mathbb{A}}^1)^{\mathrm{Spec}(R)}$ is a singleton. Hence $R = 0$.
Therefore x is invertible.

Nongeometric sequents

Let $\mathbb{T}$ be a **geometric theory** (rings, intervals, ...).

For a **geometric sequent** $\forall \vec{x}.\, (\varphi \Rightarrow \psi)$, the following are equivalent:

1. It is **provable** by $\mathbb{T}$.
2. It holds **for all models** of $\mathbb{T}$ in all toposes.
3. It holds for the **generic model** of $\mathbb{T}$ in its **classifying topos**.

- Additional **nongeometric sequents** may hold in a classifying topos, for instance "$\underline{\mathbb{A}}^1$ is synthetically quasicoherent" in $\mathrm{Zar}(S)$.
- These are **$\mathbb{T}$-redundant**, but the converse is false. [Bezem–Buchholtz–Coquand 2017; answering a question by Wraith possibly raised at PSSL 1.]
- Are they precisely the consequences of synthetic quasicoherence?
- Applications: synthetic algebraic geometry, generic freeness, ...

Further research

- Push synthetic algebraic geometry further: true cohomology, intersection theory, derived categories, …
- What do the various subtoposes of $\mathrm{Zar}(S)$ classify (étale, fppf, ph, $\neg\neg$, …)? What about the crystalline topos?
- Understand quasicoherence.
- Find further applications of nongeometric sequents, for instance in constructive algebra.

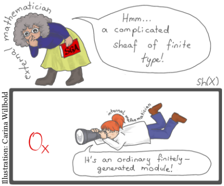


Illustration: Carina Willbold

Expository notes:

https://www.ingo-blechschmidt.eu/