Without loss of generality, any reduced ring is a field.

Interruptions welcome at any point.

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Summary

- For any reduced ring A, there is a semantics with
$$A \models \big(\forall x.\, \neg(\exists y.\, xy = 1) \Rightarrow x = 0\big).$$
- This semantics is sound with respect to intuitionistic logic.
- It has uses in classical and constructive commutative algebra.

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- This semantics is sound with respect to intuitionistic logic.
- It has uses in classical and constructive commutative algebra.

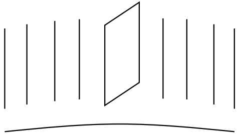
A baby application

Let M be a surjective matrix with more rows than columns over a ring A. Then $A = 0$.

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

Generic freeness

Generically, any finitely generated module over a reduced ring is free.



Motivating the semantics

A ring is **local** iff $1 \neq 0$ and $x + y = 1$ implies that x is invertible or y is invertible.

Examples: $k,\ k[[X]],\ \mathbb{C}\{z\},\ \mathbb{Z}_{(p)}$
Non-examples: $\mathbb{Z},\ k[X],\ \mathbb{Z}/(pq)$

Locally, any ring is local.

Let $x + y = 1$ in a ring A. Then:

- The element x is invertible in $A[x^{-1}]$.
- The element y is invertible in $A[y^{-1}]$.

The semantics

Let A be a fixed ring. Let "$A \models \varphi$" be a shorthand for "$1 \models \varphi$".

$$
\begin{array}{lll}
f \models \top & \text{iff} & \top \\
f \models \bot & \text{iff} & f \text{ is nilpotent} \\
f \models x = y & \text{iff} & x = y \in A[f^{-1}] \\
f \models \varphi \wedge \psi & \text{iff} & f \models \varphi \text{ and } f \models \psi \\
f \models \varphi \vee \psi & \text{iff} & \text{there exists a partition } f^n = fg_1 + \cdots + fg_m \text{ with,} \\
& & \qquad \text{for each } i,\ fg_i \models \varphi \text{ or } fg_i \models \psi \\
f \models \varphi \Rightarrow \psi & \text{iff} & \text{for all } g \in A,\ fg \models \varphi \text{ implies } fg \models \psi \\
f \models \forall x : A^\sim .\, \varphi & \text{iff} & \text{for all } g \in A \text{ and } x_0 \in A[(fg)^{-1}],\ fg \models \varphi[x_0/x] \\
f \models \exists x : A^\sim .\, \varphi & \text{iff} & \text{there exists a partition } f^n = fg_1 + \cdots + fg_m \text{ with,} \\
& & \qquad \text{for each } i,\ g_i \models \varphi[x_0/x] \text{ for some } x_0 \in A[(fg_i)^{-1}]
\end{array}
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The semantics

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$$\begin{array}{lll} f \models x = y & \text{iff} & x = y \in A[f^{-1}] \\ f \models \varphi \wedge \psi & \text{iff} & f \models \varphi \text{ and } f \models \psi \\ f \models \varphi \vee \psi & \text{iff} & \text{there exists a partition } f^n = fg_1 + \cdots + fg_m \text{ with,} \\ & & \quad \text{for each } i,\ fg_i \models \varphi \text{ or } fg_i \models \psi \end{array}$$

Monotonicity

If $f \models \varphi$, then also $fg \models \varphi$.

Locality

If $f^n = fg_1 + \cdots + fg_m$ and $fg_i \models \varphi$ for all i, then also $f \models \varphi$.

Soundness

If $\varphi \vdash \psi$ and $f \models \varphi$, then $f \models \psi$.

Forced properties

$A \models \ulcorner A^{\sim}$ is a local ring$\urcorner$.

A baby application

Let $M \in A^{n \times m}$ be a surjective matrix over a ring A. If $n > m$, then $1 = 0 \in A$.

Classical proof. Assume to the contrary that $1 \neq 0 \in A$. Pick a maximal ideal $\mathfrak{m}$ of A. Then M is surjective as a matrix over the field $A/\mathfrak{m}$. This is in contradiction to basic linear algebra. $\square$

Constructive proof. We verify that $A \models \ulcorner M$ is surjective$\urcorner$. Since the claim admits an intuitionistic proof in the case that the ring is local, soundness implies that $A \models 1 = 0$. Thus $1 = 0 \in A$. $\square$

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NONTRIVIAL USES OF TRIVIAL RINGS

FRED RICHMAN

(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. Four theorems about commutative rings are proved with the aid

Investigating the forcing model

Assuming the Boolean prime ideal theorem, any first-order formula "$\forall \ldots \forall. (\cdots \Longrightarrow \cdots)$", where the two subformulas may not contain "$\Rightarrow$" and "$\forall$", holds for $A^\sim$ iff it holds for all stalks $A_\mathfrak{p}$.

Examples: being local, reduced, an integral domain.

Investigating the forcing model

Assuming the Boolean prime ideal theorem, any first-order formula "$\forall \ldots \forall. (\cdots \Longrightarrow \cdots)$", where the two subformulas may not contain "$\Rightarrow$" and "$\forall$", holds for $A^\sim$ iff it holds for all stalks $A_\mathfrak{p}$.

Examples: being local, reduced, an integral domain.

The forcing model has additional **unique properties**, e. g.

$$A \models \forall x : A^\sim. \neg(\ulcorner x \text{ inv.} \urcorner) \Longrightarrow \ulcorner x \text{ nilpotent} \urcorner$$

which if A is reduced implies the **field condition**

$$A \models \forall x : A^\sim. \neg(\ulcorner x \text{ inv.} \urcorner) \Longrightarrow x = 0 \quad \text{and also}$$
$$A \models \forall x : A^\sim. \neg\neg(x = 0) \Longrightarrow x = 0.$$

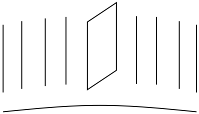
Translation. For any element $x \in A$, if $f = 0$ is the only element such that x is invertible in $A[f^{-1}]$, then $x = 0$.

Grothendieck's generic freeness

Let A be a reduced ring.
Let B be an A-algebra of finite type ($\cong A[X_1, \ldots, X_n]/\mathfrak{a}$).
Let M be a finitely generated B-module ($\cong B^m/U$).

Theorem. If $1 \neq 0$ in A, there exists $f \neq 0$ in A such that

1. $B[f^{-1}]$ and $M[f^{-1}]$ are free modules over $A[f^{-1}]$,
2. $A[f^{-1}] \to B[f^{-1}]$ is of finite presentation, and
3. $M[f^{-1}]$ is finitely presented as a module over $B[f^{-1}]$.



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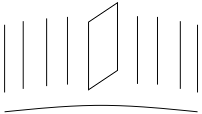

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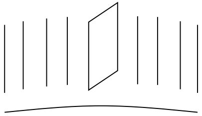

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- No generalization to unreduced rings.
- Implies the law of excluded middle.
- **Constructive restatement.** If zero is the only element $f \in A$ such that 1, 2, and 3, then $1 = 0 \in A$.

A constructive proof

Let A be a reduced ring.
Let B be an A-algebra of finite type ($\cong A[X_1, \ldots, X_n]/\mathfrak{a}$).
Let M be a finitely generated B-module ($\cong B^m/U$).

Theorem. If zero is the only element $f \in A$ such that

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then $1 = 0 \in A$.

Constructive proof. Observe that the theorem amounts to

$A \models \ulcorner$It's **not not** the case that

1. $B^\sim$ and $M^\sim$ are free modules over $A^\sim$,
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A constructive proof

Let A be a reduced ring.
Let B be an A-algebra of finite type ($\cong A[X_1, \ldots, X_n]/\mathfrak{a}$).
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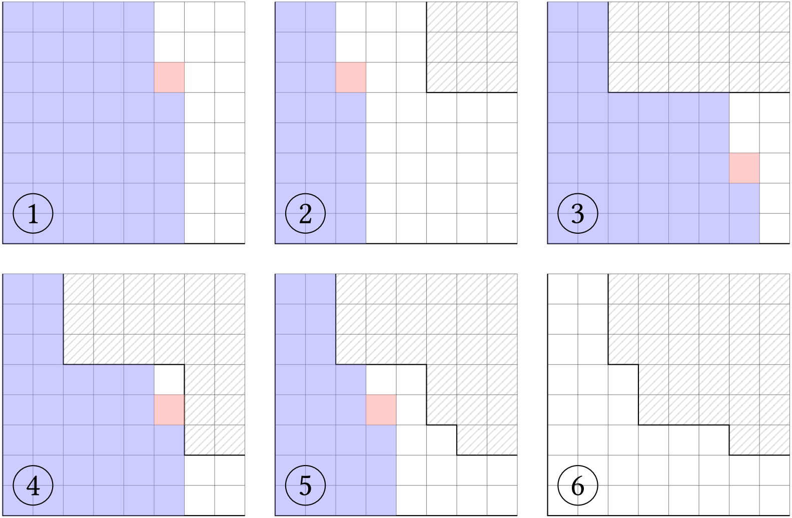
2. $A^\sim \to B^\sim$ is of finite presentation, and
3. $M^\sim$ is finitely presented as a module over $B^\sim\urcorner$.

Claims 2 and 3 follow from the fact that $A^\sim$ is **anonymously Noetherian** (any ideal is **not not** finitely generated) which entails that $A^\sim[X_1, \ldots, X_n]$ is anonymously Noetherian.

Claim 1 follows from a careful rendition of the standard linear algebra proof, employing Dickson's lemma to ensure termination. □

Assume that $B^\sim$ is generated by $(x^i y^j)_{i,j\geq 0}$ as an $A^\sim$-module. It's **not not** the case that either some generator can be expressed as a linear combination of others with smaller index, or not.



An explicit constructive proof

Lemma. Let A be a ring. Let M be an A-module with generating family $(x_1, \ldots, x_n)$. Assume that the only element $g \in A$ such that one of the x_i is an $A[g^{-1}]$-linear combination in $A[g^{-1}]$ of the other generators is $g = 0$. Then M is free with $(x_1, \ldots, x_n)$ as a basis.

Proof. Let $\sum_i a_i x_i = 0$. Let i be arbitrary. In $M[a_i^{-1}]$, the generator x_i is a linear combination of the other generators. Thus $a_i = 0$. □

Theorem. Let A be a reduced ring. Let M be a finitely generated A-module. If zero is the only element $f \in A$ such that $M[f^{-1}]$ is finite free as an $A[f^{-1}]$-module, then $1 = 0$ in A.

Proof. By induction on the length n of a generating family $(x_1, \ldots, x_n)$ of M.

We verify the assumption of the lemma. Thus let $g \in A$ be given such that one of the x_i is an $A[g^{-1}]$-linear combination of the others in $M[g^{-1}]$. Therefore the $A[g^{-1}]$-module $M[g^{-1}]$ can be generated by $n - 1$ elements. By the induction hypothesis (applied to $A[g^{-1}]$ and its module $M[g^{-1}]$) it follows that $A[g^{-1}] = 0$. Therefore $g = 0$.

Thus M is free. We finish by using the assumption for $f = 1$. □

An explicit constructive proof

Theorem. Let A be a reduced ring. Let B be a finitely generated A-algebra. If zero is the only element $f \in A$ such that $B[f^{-1}]$ is finitely presented as an $A[f^{-1}]$-algebra, then $1 = 0$ in A.

Proof. Write $B = A[X_1, \ldots, X_n]/\mathfrak{a}$. We describe only the case $n = 0$.

As a first step, we verify $\mathfrak{a} = (0)$. Let $f \in \mathfrak{a}$. Then $B[f^{-1}] = 0$. Thus $f = 0$ by assumption.

We now use the assumption again, this time for $f = 1$. $\square$