



Exploring mathematical objects from custom-tailored mathematical universes

– *an invitation* –

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Third international conference of the
Italian Network for the Philosophy of Mathematics
in Mussomeli

May 26th, 2018

A glimpse of the toposophic landscape

Set



The usual laws
of logic hold.

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Sh X



The axiom of
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Eff



Every function
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The internal universe of a topos

For any topos \mathcal{E} and any statement φ , we define the meaning of

$\mathcal{E} \models \varphi$ (“ φ holds in the internal universe of \mathcal{E} ”)

using the **Kripke–Joyal semantics**.

$\mathbf{Set} \models \varphi$

“ φ holds in the
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$\mathbf{Sh}(X) \models \varphi$

“ φ holds
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no $\varphi \vee \neg\varphi$, no $\neg\neg\varphi \Rightarrow \varphi$, no axiom of choice

First steps in alternate universes

- $\text{Eff} \models$ “Any number is prime or is not prime.” ✓
Meaning: There is a **Turing machine** which determines of any given number whether it is prime or not.
- $\text{Eff} \models$ “There are infinitely many prime numbers.” ✓
Meaning: There is a **Turing machine** producing arbitrarily many primes.
- $\text{Eff} \models$ “Any function $\mathbb{N} \rightarrow \mathbb{N}$ is the zero function or not.” ✗
Meaning: There is a **Turing machine** which, given a Turing machine computing a function $f : \mathbb{N} \rightarrow \mathbb{N}$, determines whether f is zero or not.
- $\text{Eff} \models$ “Any function $\mathbb{N} \rightarrow \mathbb{N}$ is computable.” ✓
- $\text{Sh}(X) \models$ “Any cont. function with opposite signs has a zero.” ✗
Meaning: Zeros can locally be picked **continuously** in continuous families of continuous functions. (video for counterexample)

Applications in commutative algebra

Let A be a reduced commutative ring.

For instance: \mathbb{Z} , $\mathbb{Z}[X]$, $\mathbb{Z}[X, Y, Z]/(X^n + Y^n - Z^n)$, \mathbb{Q} , \mathbb{R}

The **little Zariski topos** of A contains a **mirror image** of A : A^\sim .

1 A^\sim is always a **field**.

2 A^\sim is still **very close** to A .

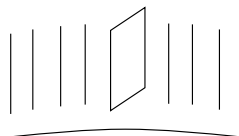
A baby application

Let M be a surjective matrix with more rows than columns over a ring A . Then $A = 0$.

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

Generic freeness

Generically, any finitely generated module over a reduced ring is free.



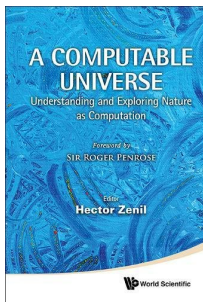
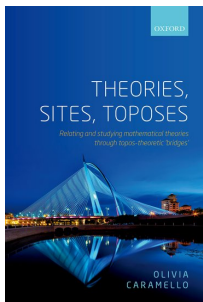
The little Zariski topos in more detail

Recall $A[f^{-1}] = \left\{ \frac{u}{f^n} \mid u \in A, n \in \mathbb{N} \right\}$.

- $\text{Sh}(\text{Spec}(A)) \models \text{“For all } x \in A^\sim, \dots\text{”}$
Meaning: For all $f \in A$ and all $x \in A[f^{-1}]$, ...
- $\text{Sh}(\text{Spec}(A)) \models \text{“There is } x \in A^\sim \text{ such that } \dots\text{”}$
Meaning: There is a partition of unity, $1 = f_1 + \dots + f_n \in A$, such that for each i , there exists $x_i \in A[f_i^{-1}]$ with ...
- $\text{Sh}(\text{Spec}(A)) \models \text{“}\varphi \text{ implies } \psi\text{”}$
Meaning: For all $f \in A$, if φ on stage f , then ψ on stage f .

Topos theory ...

- enriches the platonism debate,
- uncovers further relations between objects,
- allows to study objects from a different point of view,
- has applications in mathematical practice.



Spiel und Spaß mit der internen Welt des kleinen Zariski-Topos

Ingo Bloeschmidt

19. Dezember 2013



$R \models x = y : \mathcal{O}$	Für die gegebenen Elemente $x, y \in R$ gilt $x = y$.
$R \models \top$	$1 \in R$. (Das ist stets erfüllt.)
$R \models 0$	$0 \in R$. (Das ist gerade in Nullkörpern erfüllt.)
$R \models \rho \wedge \nu$	$R \models \rho$ und $R \models \nu$.
$R \models \rho \vee \nu$	$R \models \rho$ oder $R \models \nu$.
$R \models \rho \Rightarrow \nu$	Es gibt eine Zerlegung $\sum_{i=1}^n 1 \in R$ sodass für alle i jeweils $R \models \rho \Rightarrow \nu$ oder $R \models \nu$.
$R \models \rho \Leftrightarrow \nu$	Für jedes $x \in R$ gilt: $R \models \rho \Leftrightarrow R \models \nu$ oder $R \models \nu$.
$R \models \forall x : \mathcal{O}$	Für jedes $x \in R$ und jedes $\nu \in R[x] : R \models \nu(x)$.
$R \models \exists x : \mathcal{O}$	Es gibt eine Zerlegung $\sum_{i=1}^n 1 \in R$ für alle i Elemente $x_i \in R[x_i] : 1 \in R$ und $R \models \nu(x_i)$.

Die Kinder-Jugend-Semantik des Kleinen Zariski-Torus

Using the internal language of toposes
in algebraic geometry

Dissertation
 zur Erlangung des akademischen Grades

Dr. rer. nat.

eingereicht an der

Mathematisch-Naturwissenschaftlich-Technischen Fakultät
der Universität Augsburg

1994, 1995, 1996, 1997, 1998, 1999, 2000, 2001, 2002, 2003, 2004, 2005, 2006, 2007, 2008, 2009, 2010, 2011, 2012, 2013, 2014, 2015, 2016, 2017, 2018, 2019, 2020, 2021, 2022, 2023, 2024, 2025, 2026, 2027, 2028, 2029, 2030, 2031, 2032, 2033, 2034, 2035, 2036, 2037, 2038, 2039, 2040, 2041, 2042, 2043, 2044, 2045, 2046, 2047, 2048, 2049, 2050, 2051, 2052, 2053, 2054, 2055, 2056, 2057, 2058, 2059, 2060, 2061, 2062, 2063, 2064, 2065, 2066, 2067, 2068, 2069, 2070, 2071, 2072, 2073, 2074, 2075, 2076, 2077, 2078, 2079, 2080, 2081, 2082, 2083, 2084, 2085, 2086, 2087, 2088, 2089, 2090, 2091, 2092, 2093, 2094, 2095, 2096, 2097, 2098, 2099, 2100, 2101, 2102, 2103, 2104, 2105, 2106, 2107, 2108, 2109, 2110, 2111, 2112, 2113, 2114, 2115, 2116, 2117, 2118, 2119, 2120, 2121, 2122, 2123, 2124, 2125, 2126, 2127, 2128, 2129, 2130, 2131, 2132, 2133, 2134, 2135, 2136, 2137, 2138, 2139, 2140, 2141, 2142, 2143, 2144, 2145, 2146, 2147, 2148, 2149, 2150, 2151, 2152, 2153, 2154, 2155, 2156, 2157, 2158, 2159, 2160, 2161, 2162, 2163, 2164, 2165, 2166, 2167, 2168, 2169, 2170, 2171, 2172, 2173, 2174, 2175, 2176, 2177, 2178, 2179, 2180, 2181, 2182, 2183, 2184, 2185, 2186, 2187, 2188, 2189, 2190, 2191, 2192, 2193, 2194, 2195, 2196, 2197, 2198, 2199, 2200, 2201, 2202, 2203, 2204, 2205, 2206, 2207, 2208, 2209, 2210, 2211, 2212, 2213, 2214, 2215, 2216, 2217, 2218, 2219, 2220, 2221, 2222, 2223, 2224, 2225, 2226, 2227, 2228, 2229, 2230, 2231, 2232, 2233, 2234, 2235, 2236, 2237, 2238, 2239, 2240, 2241, 2242, 2243, 2244, 2245, 2246, 2247, 2248, 2249, 2250, 2251, 2252, 2253, 2254, 2255, 2256, 2257, 2258, 2259, 2260, 2261, 2262, 2263, 2264, 2265, 2266, 2267, 2268, 2269, 2270, 2271, 2272, 2273, 2274, 2275, 2276, 2277, 2278, 2279, 2280, 2281, 2282, 2283, 2284, 2285, 2286, 2287, 2288, 2289, 2290, 2291, 2292, 2293, 2294, 2295, 2296, 2297, 2298, 2299, 2300, 2301, 2302, 2303, 2304, 2305, 2306, 2307, 2308, 2309, 2310, 2311, 2312, 2313, 2314, 2315, 2316, 2317, 2318, 2319, 2320, 2321, 2322, 2323, 2324, 2325, 2326, 2327, 2328, 2329, 2330, 2331, 2332, 2333, 2334, 2335, 2336, 2337, 2338, 2339, 2340, 2341, 2342, 2343, 2344, 2345, 2346, 2347, 2348, 2349, 2350, 2351, 2352, 2353, 2354, 2355, 2356, 2357, 2358, 2359, 2360, 2361, 2362, 2363, 2364, 2365, 2366, 2367, 2368, 2369, 2370, 2371, 2372, 2373, 2374, 2375, 2376, 2377, 2378, 2379, 2380, 2381, 2382, 2383, 2384, 2385, 2386, 2387, 2388, 2389, 2390, 2391, 2392, 2393, 2394, 2395, 2396, 2397, 2398, 2399, 2400, 2401, 2402, 2403, 2404, 2405, 2406, 2407, 2408, 2409, 2410, 2411, 2412, 2413, 2414, 2415, 2416, 2417, 2418, 2419, 2420, 2421, 2422, 2423, 2424, 2425, 2426, 2427, 2428, 2429, 2430, 2431, 2432, 2433, 2434, 2435, 2436, 2437, 2438, 2439, 2440, 2441, 2442, 2443, 2444, 2445, 2446, 2447, 2448, 2449, 2450, 2451, 2452, 2453, 2454, 2455, 2456, 2457, 2458, 2459, 2460, 2461, 2462, 2463, 2464, 2465, 2466, 2467, 2468, 2469, 2470, 2471, 2472, 2473, 2474, 2475, 2476, 2477, 2478, 2479, 2480, 2481, 2482, 2483, 2484, 2485, 2486, 2487, 2488, 2489, 2490, 2491, 2492, 2493, 2494, 2495, 2496, 2497, 2498, 2499, 2500, 2501, 2502, 2503, 2504, 2505, 2506, 2507, 2508, 2509, 2510, 2511, 2512, 2513, 2514, 2515, 2516, 2517, 2518, 2519, 2520, 2521, 2522, 2523, 2524, 2525, 2526, 2527, 2528, 2529, 2530, 2531, 2532, 2533, 2534, 2535, 2536, 2537, 2538, 2539, 2540, 2541, 2542, 2543, 2544, 2545, 2546, 2547, 2548, 2549, 2550, 2551, 2552, 2553, 2554, 2555, 2556, 2557, 2558, 2559, 2560, 2561, 2562, 2563, 2564, 2565, 2566, 2567, 2568, 2569, 2570, 2571, 2572, 2573, 2574, 2575, 2576, 2577, 2578, 2579, 2580, 2581, 2582, 2583, 2584, 2585, 2586, 2587, 2588, 2589, 2590, 2591, 2592, 2593, 2594, 2595, 2596, 2597, 2598, 2599, 2600, 2601, 2602, 2603, 2604, 2605, 2606, 2607, 2608, 2609, 2610, 2611, 2612, 2613, 2614, 2615, 2616, 2617, 2618, 2619, 2620, 2621, 2622, 2623, 2624, 2625, 2626, 2627, 2628, 2629, 2630, 2631, 2632, 2633, 2634, 2635, 2636, 2637, 2638, 2639, 2640, 2641, 2642, 2643, 2644, 2645, 2646, 2647, 2648, 2649, 2650, 2651, 2652, 2653, 2654, 2655, 2656, 2657, 2658, 2659, 2660, 2661, 2662, 2663, 2664, 2665, 2666, 2667, 2668, 2669, 2670, 2671, 2672, 2673, 2674, 2675, 26

From: Elizabeth Smith



June 2017