Exploring mathematical objects from custom-tailored mathematical universes

– an invitation –

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A glimpse of the toposophic landscape

Set

The usual laws of logic hold.

A glimpse of the toposophic landscape

Set

The usual laws of logic hold.

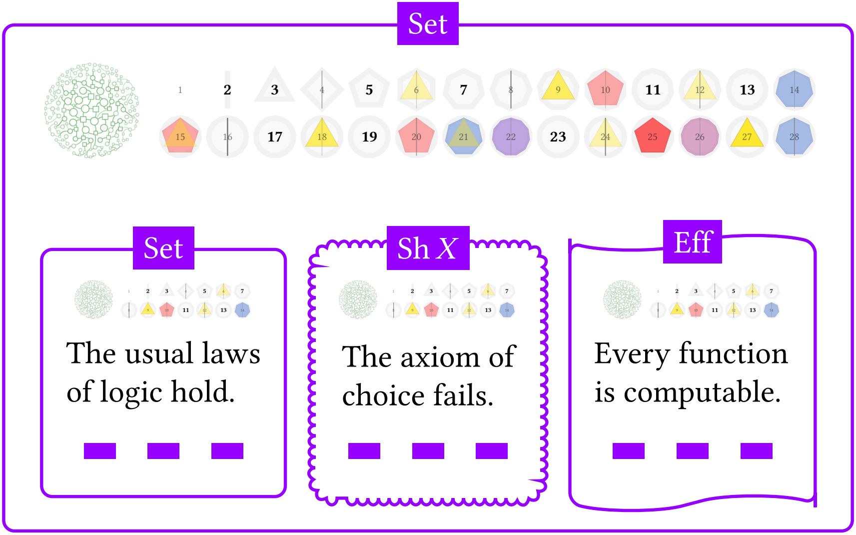
Sh X

The axiom of choice fails.

Eff

Every function is computable.

A glimpse of the toposophic landscape



The internal universe of a topos

For any topos $\mathcal{E}$ and any statement φ, we define the meaning of

$$\text{“}\mathcal{E} \models \varphi\text{”} \quad (\text{“}\varphi \text{ holds in the internal universe of } \mathcal{E}\text{”})$$

using the **Kripke–Joyal semantics**.

$\mathrm{Set} \models \varphi$
“φ holds in the usual sense.”

$\mathrm{Sh}(X) \models \varphi$
“φ holds continuously.”

$\mathrm{Eff} \models \varphi$
“φ holds computably.”

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Any topos supports **mathematical reasoning**:

If $\mathcal{E} \models \varphi$ and if $\varphi \vdash \psi$ intuitionistically, then $\mathcal{E} \models \psi$.

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no $\varphi \vee \neg\varphi$, no $\neg\neg\varphi \Rightarrow \varphi$, no axiom of choice

First steps in alternate universes

- Eff $\models$ "Any number is prime or is not prime." ✓
  Meaning: There is a **Turing machine** which determines of any given number whether it is prime or not.

- Eff $\models$ "There are infinitely many prime numbers." ✓
  Meaning: There is a **Turing machine** producing arbitrarily many primes.

- Eff $\models$ "Any function $\mathbb{N} \to \mathbb{N}$ is the zero function or not." ✗
  Meaning: There is a **Turing machine** which, given a Turing machine computing a function $f : \mathbb{N} \to \mathbb{N}$, determines whether f is zero or not.

- Eff $\models$ "Any function $\mathbb{N} \to \mathbb{N}$ is computable." ✓

- $\mathrm{Sh}(X) \models$ "Any cont. function with opposite signs has a zero." ✗
  Meaning: Zeros can locally be picked **continuously** in continuous families of continuous functions. (video for counterexample)

Applications in commutative algebra

Let A be a reduced commutative ring.

For instance: $\mathbb{Z},\ \mathbb{Z}[X],\ \mathbb{Z}[X,Y,Z]/(X^n+Y^n-Z^n),\ \mathbb{Q},\ \mathbb{R}$

The **little Zariski topos** of A contains a **mirror image** of A: $A^\sim$.

1. $A^\sim$ is always a **field**.
2. $A^\sim$ is still **very close** to A.

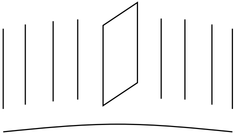
A baby application

Let M be a surjective matrix with more rows than columns over a ring A. Then $A = 0$.

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

Generic freeness

Generically, any finitely generated module over a reduced ring is free.



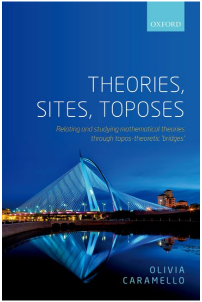
The little Zariski topos in more detail

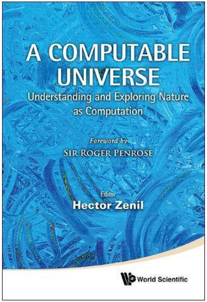
Recall $A[f^{-1}] = \{ \frac{u}{f^n} \mid u \in A, n \in \mathbb{N} \}$.

- $\mathrm{Sh}(\mathrm{Spec}(A)) \models$ "For all $x \in A^\sim$, …"
  Meaning: For all $f \in A$ and all $x \in A[f^{-1}]$, …
- $\mathrm{Sh}(\mathrm{Spec}(A)) \models$ "There is $x \in A^\sim$ such that …"
  Meaning: There is a partition of unity, $1 = f_1 + \cdots + f_n \in A$, such that for each i, there exists $x_i \in A[f_i^{-1}]$ with …
- $\mathrm{Sh}(\mathrm{Spec}(A)) \models$ "φ implies ψ"
  Meaning: For all $f \in A$, if φ on stage f, then ψ on stage f.

Topos theory …

- enriches the platonism debate,
- uncovers further relations between objects,
- allows to study objects from a different point of view,
- has applications in mathematical practice.





Spiel und Spaß mit der internen Welt des kleinen Zariski-Topos

Ingo Blechschmidt

19. Dezember 2013

$R \models x = y : \mathcal{O}$	$:\Longleftrightarrow$	Für die gegebenen Elemente $x, y \in R$ gilt $x = y$.
$R \models \top$	$:\Longleftrightarrow$	$1 = 1 \in R$. (Das ist stets erfüllt.)
$R \models \bot$	$:\Longleftrightarrow$	$1 = 0 \in R$. (Das ist genau in Nullringen erfüllt.)
$R \models \phi \wedge \psi$	$:\Longleftrightarrow$	$R \models \phi$ und $R \models \psi$.
$R \models \phi \vee \psi$	$:\Longleftrightarrow$	~~$R \models \phi$ oder $R \models \psi$.~~
$R \models \phi \vee \psi$	$:\Longleftrightarrow$	Es gibt eine Zerlegung $\sum_i s_i = 1 \in R$ sodass für alle i jeweils $R[s_i^{-1}] \models \phi$ oder $R[s_i^{-1}] \models \psi$.
$R \models \phi \Rightarrow \psi$	$:\Longleftrightarrow$	Für jedes $s \in R$ gilt: Aus $R[s^{-1}] \models \phi$ folgt $R[s^{-1}] \models \psi$.
$R \models \forall x : \mathcal{O}.\ \phi$	$:\Longleftrightarrow$	Für jedes $s \in R$ und jedes $x \in R[s^{-1}]$ gilt: $R[s^{-1}] \models \phi(x)$.
$R \models \exists x : \mathcal{O}.\ \phi$	$:\Longleftrightarrow$	Es gibt eine Zerlegung $\sum_i s_i = 1 \in R$ und Elemente $x_i \in R[s_i^{-1}]$ sodass für alle i: $R[s_i^{-1}] \models \phi(x_i)$.

Die Kripke–Joyal-Semantik des kleinen Zariski-Topos.

Using the internal language of toposes in algebraic geometry

Dissertation
zur Erlangung des akademischen Grades

Dr. rer. nat.

eingereicht an der

Mathematisch-Naturwissenschaftlich-Technischen Fakultät
der Universität Augsburg

von

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