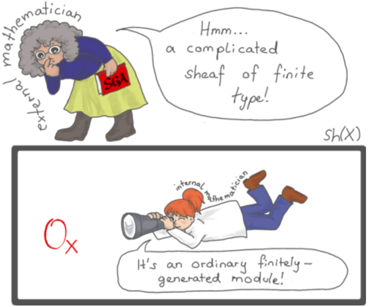

First steps in synthetic algebraic geometry

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Camping at Unterbacher See

Quick summary

By employing the internal language of toposes in various ways, you can pretend that:

1. Reduced rings are Noetherian and in fact fields.
2. Sheaves of modules are plain modules.
3. Schemes are sets:

$$\mathbb{P}^2_S = \{[x_0 : x_1 : x_2] \mid x_0 \neq 0 \vee x_1 \neq 0 \vee x_2 \neq 0\}.$$

What is a topos?

Formal definition

A **topos** is a category which has finite limits, is cartesian closed and has a subobject classifier.

Motto

A topos is a category which is sufficiently rich to support an **internal language**.

Examples

- Set: category of sets
- $\mathrm{Sh}(X)$: category of set-valued sheaves on a space X
- $\mathrm{Zar}(S)$: big Zariski topos of a base scheme S

What is the internal language?

The internal language of a topos $\mathcal{E}$ allows to

1. construct objects and morphisms of the topos,
2. formulate statements about them and
3. prove such statements

in a **naive element-based** language:

externally	internally to $\mathcal{E}$
object of $\mathcal{E}$	set
morphism in $\mathcal{E}$	map of sets
monomorphism	injective map
epimorphism	surjective map
group object	group

The internal language of Sh(X)

Let X be a topological space. Then we recursively define

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ.

The internal language of $\mathrm{Sh}(X)$

Let X be a topological space. Then we recursively define

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ.

$$U \models f = g : \mathcal{F} \quad :\Longleftrightarrow f|_U = g|_U \in \mathcal{F}(U)$$

$$U \models \varphi \wedge \psi \quad :\Longleftrightarrow U \models \varphi \text{ and } U \models \psi$$

$$U \models \varphi \vee \psi \quad :\Longleftrightarrow \sout{U \models \varphi \text{ or } U \models \psi}$$

there exists a covering $U = \bigcup_i U_i$ s. th. for all i:

$$U_i \models \varphi \text{ or } U_i \models \psi$$

$$U \models \varphi \Rightarrow \psi \quad :\Longleftrightarrow \text{for all open } V \subseteq U\text{: } V \models \varphi \text{ implies } V \models \psi$$

$$U \models \forall f \colon \mathcal{F}.\, \varphi(f) :\Longleftrightarrow \text{for all sections } f \in \mathcal{F}(V), V \subseteq U\text{: } V \models \varphi(f)$$

$$U \models \exists f \colon \mathcal{F}.\, \varphi(f) :\Longleftrightarrow \text{there exists a covering } U = \textstyle\bigcup_i U_i \text{ s. th. for all } i\text{:}$$

there exists $f_i \in \mathcal{F}(U_i)$ s. th. $U_i \models \varphi(f_i)$

The internal language of Sh(X)

Crucial property: Locality

If $U = \bigcup_i U_i$, then $U \models \varphi$ iff $U_i \models \varphi$ for all i.

Crucial property: Soundness

If $U \models \varphi$ and if φ implies ψ constructively, then $U \models \psi$.

The internal language of $\mathrm{Sh}(X)$

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A first glance at the constructive nature

- $U \models f = 0$ iff $f|_U = 0 \in \Gamma(U, \mathcal{F})$.
- $U \models \neg\neg(f = 0)$ iff $f = 0$ on a dense open subset of U.

The little Zariski topos

Definition

The **little Zariski topos** of a scheme X is the category $\mathrm{Sh}(X)$ of set-valued sheaves on X.

- Internally, the structure sheaf $\mathcal{O}_X$ looks like

  an ordinary ring.

- Internally, a sheaf of $\mathcal{O}_X$-modules looks like

  an ordinary module on that ring.

Building a dictionary

Understand notions of algebraic geometry as notions of algebra internal to $\mathrm{Sh}(X)$.

externally	internally to $\mathrm{Sh}(X)$
sheaf of sets	set
morphism of sheaves	map of sets
monomorphism	injective map
epimorphism	surjective map
sheaf of rings	ring
sheaf of modules	module
sheaf of finite type	finitely generated module
finite locally free sheaf	finite free module
tensor product of sheaves	tensor product of modules
sheaf of Kähler differentials	module of Kähler differentials
dimension of X	Krull dimension of $\mathcal{O}_X$

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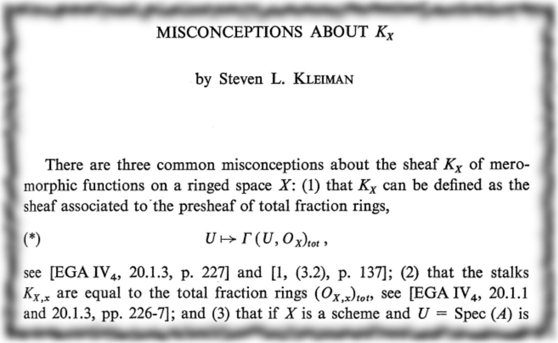
Understand notions of algebraic geometry as notions of algebra internal to $\mathrm{Sh}(X)$.



MISCONCEPTIONS ABOUT K_X

by Steven L. KLEIMAN

There are three common misconceptions about the sheaf K_X of meromorphic functions on a ringed space X: (1) that K_X can be defined as the sheaf associated to the presheaf of total fraction rings,

(*) $$U \mapsto \Gamma(U, O_X)_{tot},$$

see [EGA IV$_4$, 20.1.3, p. 227] and [1, (3.2), p. 137]; (2) that the stalks $K_{X,x}$ are equal to the total fraction rings $(O_{X,x})_{tot}$, see [EGA IV$_4$, 20.1.1 and 20.1.3, pp. 226-7]; and (3) that if X is a scheme and $U = \mathrm{Spec}\,(A)$ is

externally	internally to $\mathrm{Sh}(X)$
sheaf of Kähler differentials	module of Kähler differentials
dimension of X	Krull dimension of $\mathcal{O}_X$

Using the dictionary

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If M' and M'' are finitely generated, so is M.

$$\Downarrow$$

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are of finite type, so is $\mathcal{F}$.

Using the dictionary

Any finitely generated vector space does *not not* possess a basis.

$$\Downarrow$$

Any sheaf of modules of finite type on a reduced scheme is locally free *on a dense open subset.*

Ravi Vakil: "Important hard exercise" (13.7.K).

The objective

Understand notions and statements of **algebraic geometry** as notions and statements of (constructive) **commutative algebra** internal to the **little Zariski topos**.

Further topics regarding the little Zariski topos:

- Unique features of the internal world
- Transfer principles $M \leftrightarrow M^\sim$
- The curious role of affine open subsets
- Quasicoherence
- Spreading from points to neighbourhoods
- The relative spectrum

Unique features of the internal world

Let X be a scheme. Internally to $\mathrm{Sh}(X)$,

any non-invertible element of $\mathcal{O}_X$ is nilpotent.

> ON THE SPECTRUM OF A RINGED TOPOS 209
>
> For completeness, two further remarks should be added to this treatment of the spectrum. One is that in **E** the canonical map $A \to \Gamma_*(LA)$ is an isomorphism—i.e., the representation of A in the ring of "global sections" of LA is complete. The second, due to Mulvey in the case $\mathbf{E} = \mathbf{S}$, is that in Spec(**E**, A) the formula
>
> $$\neg(x \in U(LA)) \Rightarrow \exists n(x^n = 0)$$
>
> is valid. This is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas. In any case, calculations such as these are easier from the point of view of the Heyting algebra of radical ideals of A, and hence will be omitted here.

Miles Tierney. On the spectrum of a ringed topos. 1976.

Unique features of the internal world

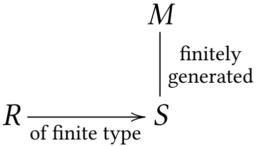
Let X be a scheme. Internally to $\mathrm{Sh}(X)$,

any non-invertible element of $\mathcal{O}_X$ is nilpotent.

If X is reduced, this implies:

- $\mathcal{O}_X$ is a **field** in that $\neg(x \text{ invertible}) \Rightarrow x = 0$.
- $\mathcal{O}_X$ has **$\neg\neg$-stable equality**: $\neg\neg(x = 0) \Rightarrow x = 0$.
- $\mathcal{O}_X$ is **anonymously Noetherian**: Any ideal of $\mathcal{O}_X$ is **not not** finitely generated.

Generic freeness



Lemma. If R is reduced, there is $f \neq 0$ in R such that

1. $S[f^{-1}]$ and $M[f^{-1}]$ are free modules over $R[f^{-1}]$,
2. $R[f^{-1}] \to S[f^{-1}]$ is of finite presentation, and
3. $M[f^{-1}]$ is finitely presented as a module over $S[f^{-1}]$.

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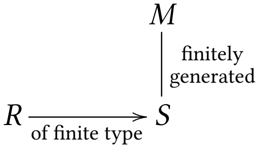

Lemma. If R is reduced, there is $f \neq 0$ in R such that

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3. $M[f^{-1}]$ is finitely presented as a module over $S[f^{-1}]$.

For a trivial proof, employ $\mathrm{Sh}(\mathrm{Spec}\, R)$ and exploit that $\mathcal{O}_{\mathrm{Spec}\, R}$ is a **field** and **anonymously Noetherian**.

Transfer principles

Question: How do the properties of

- an A-module M in Set and
- the $\mathcal{O}_X$-module $M^\sim$ in $\mathrm{Sh}(X)$, where $X = \operatorname{Spec} A$, relate?

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Observation: $M^\sim = \underline{M}[\mathcal{F}^{-1}]$, where

- $\underline{M}$ is the constant sheaf with stalks M on X and
- $\mathcal{F} \hookrightarrow \underline{A}$ is the **generic prime filter** with stalk $A \setminus \mathfrak{p}$ at $\mathfrak{p} \in \operatorname{Spec} A$.

Note: M and $\underline{M}$ share all first-order properties.

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Note: M and $\underline{M}$ share all first-order properties.

Answer: $M^\sim$ inherits those properties of M which are **stable under localisation**.

Synthetic algebraic geometry

Usual approach to algebraic geometry: **layer schemes above ordinary set theory** using either

- locally ringed spaces

  set of prime ideals of $\mathbb{Z}[X, Y, Z]/(X^n + Y^n - Z^n)$ + Zariski topology + structure sheaf

- or Grothendieck's functor-of-points account, where a scheme is a functor Ring $\to$ Set.

$$A \longmapsto \{(x, y, z) : A^3 \mid x^n + y^n - z^n = 0\}$$

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Synthetic approach: model schemes **directly as sets** in a certain nonclassical set theory.

$$\{(x, y, z) : (\underline{\mathbb{A}}^1)^3 \mid x^n + y^n - z^n = 0\}$$

The big Zariski topos

Definition

The **big Zariski topos** $\mathrm{Zar}(S)$ of a scheme S is the category $\mathrm{Sh}(\mathrm{Aff}/S)$. It consists of functors $(\mathrm{Aff}/S)^{\mathrm{op}} \to \mathrm{Set}$ satisfying the gluing condition that

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{j,k} F(U_j \cap U_k)$$

is a limit diagram for any affine scheme $U = \bigcup_i U_i$ over S.

- For an S-scheme X, its functor of points $\underline{X} = \mathrm{Hom}_S(\cdot, X)$ is an object of $\mathrm{Zar}(S)$. It feels like **the set of points** of X.
- Internally, $\underline{\mathbb{A}}^1$ (given by $\underline{\mathbb{A}}^1(X) = \mathcal{O}_X(X)$) looks like a field:

$$\mathrm{Zar}(S) \models \forall x : \underline{\mathbb{A}}^1.\ x \neq 0 \Longrightarrow x \text{ invertible}$$

Synthetic constructions

- $\mathbb{P}^n = \{(x_0, \ldots, x_n) : (\underline{\mathbb{A}}^1)^{n+1} \mid x_0 \neq 0 \vee \cdots \vee x_n \neq 0\}/(\underline{\mathbb{A}}^1)^\times$ $\cong$ set of one-dimensional subspaces of $(\underline{\mathbb{A}}^1)^{n+1}$.
  - $\mathcal{O}(1) = (\ell^\vee)_{\ell:\mathbb{P}^n}$
  - $\mathcal{O}(-1) = (\ell)_{\ell:\mathbb{P}^n}$
  - Euler sequence: $0 \to \ell^\perp \to ((\underline{\mathbb{A}}^1)^{n+1})^\vee \to \ell^\vee \to 0$

- $\operatorname{Spec} R = \operatorname{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1)}(R, \underline{\mathbb{A}}^1) =$ set of $\underline{\mathbb{A}}^1$-valued points of R.
  - $\operatorname{Spec} \underline{\mathbb{A}}^1[X, Y, Z]/(X^n + Y^n - Z^n) \cong \{(x, y, z) : (\underline{\mathbb{A}}^1)^3 \mid x^n + y^n - z^n = 0\}$
  - $\Delta := \operatorname{Spec} \underline{\mathbb{A}}^1[\varepsilon]/(\varepsilon^2) \cong \{\varepsilon : \underline{\mathbb{A}}^1 \mid \varepsilon^2 = 0\}$

- $TX = \operatorname{Hom}(\Delta, X)$.

Synthetic formulation of properties

- $\operatorname{Spec} R = \operatorname{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1)}(R, \underline{\mathbb{A}}^1) =$ set of $\underline{\mathbb{A}}^1$-valued points of R.
- An $\underline{\mathbb{A}}^1$-module E is **quasicoherent** if and only if

$$E \otimes_{\underline{\mathbb{A}}^1} R \longrightarrow \operatorname{Hom}(\operatorname{Spec} R, E)$$

  is an isomorphism for all finitely presented $\underline{\mathbb{A}}^1$-algebras R.
- In particular, any map $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is given by a polynomial.

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- A subset $U \subseteq X$ is qc-open if and only if for any $x : X$ there exist $f_1, \ldots, f_n : \underline{\mathbb{A}}^1$ such that $x \in U \Longleftrightarrow \exists i.\, f_i \neq 0$.
- Open subsets are $\neg\neg$-stable: $\neg\neg(x \in U) \Longrightarrow x \in U$.
- If $\gamma : \Delta \to X$ is a tangent vector with $\gamma(0) \in U$, then $\gamma(\varepsilon) \in U$ for all $\varepsilon \in \Delta$.

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- If $\gamma : \Delta \to X$ is a tangent vector with $\gamma(0) \in U$, then $\gamma(\varepsilon) \in U$ for all $\varepsilon \in \Delta$.

- X is separated if and only if for any $x, y : X$, there exists a quasicoherent ideal $\mathcal{J} \subseteq \underline{\mathbb{A}}^1$ such that $x = y \Longleftrightarrow \mathcal{J} = (0)$.

Semi-open and open tasks

- Do cohomology in the little Zariski topos; exploit that higher direct images look like ordinary sheaf cohomology from the internal point of view.
- Do cohomology in the big Zariski topos.
- Understand subtoposes of the big Zariski topos.

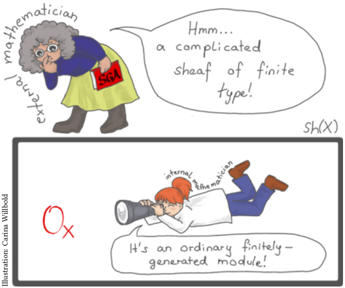
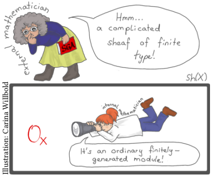


Illustration: Carina Willbold



Understand notions and statements of algebraic geometry as notions and statements of algebra internal to the little Zariski topos.

Develop a synthetic account of algebraic geometry.

- Simplify proofs and gain conceptual understanding.
- Understand relative geometry as absolute geometry.
- Contribute to constructive algebra.

http://tiny.cc/topos-notes

Participants of Augsburg's maths camp

The sun as seen from our high-altitude balloon

Translating internal statements I

Let X be a topological space (or locale) and let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then:

$$\mathrm{Sh}(X) \models \ulcorner \alpha \text{ is surjective} \urcorner$$

$$\iff \mathrm{Sh}(X) \models \forall t : \mathcal{G}.\, \exists s : \mathcal{F}.\, \alpha(s) = t$$

$\iff$ for all open $U \subseteq X$, sections $t \in \mathcal{G}(U)$:

there exists an open covering $U = \bigcup_i U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that:

$$\alpha_{U_i}(s_i) = t|_{U_i}$$

$\iff$ α is an epimorphism of sheaves

Translating internal statements II

Let X be a topological space (or locale) and let $s, t \in \mathcal{F}(X)$ be global sections of a sheaf $\mathcal{F}$ on X. Then:

$$\mathrm{Sh}(X) \models \neg\neg(s = t)$$

$$\iff \mathrm{Sh}(X) \models ((s = t) \Rightarrow \bot) \Rightarrow \bot$$

$\iff$ for all open $U \subseteq X$ such that

for all open $V \subseteq U$ such that

$s|_V = t|_V$,

it holds that $V = \emptyset$,

it holds that $U = \emptyset$

$\iff$ there exists a dense open set $W \subseteq X$ such that $s|_W = t|_W$

Spreading from points to neighbourhoods

All of the following lemmas have a short, sometimes trivial proof. Let $\mathcal{F}$ be a sheaf of finite type on a ringed space X. Let $x \in X$. Let $A \subseteq X$ be a closed subset. Then:

1. $\mathcal{F}_x = 0$ iff $\mathcal{F}|_U = 0$ for some open neighbourhood of x.
2. $\mathcal{F}|_A = 0$ iff $\mathcal{F}|_U = 0$ for some open set containing A.
3. $\mathcal{F}_x$ can be generated by n elements iff this is true on some open neighbourhood of x.
4. $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ if $\mathcal{F}$ is of finite presentation around x.
5. $\mathcal{F}$ is torsion iff $\mathcal{F}_\xi$ vanishes (assume X integral and $\mathcal{F}$ quasicoherent).

The smallest dense sublocale

Let X be a reduced scheme satisfying a technical condition. Let $i : X_{\neg\neg} \to X$ be the inclusion of the smallest dense sublocale of X.

Then $i_* i^{-1} \mathcal{O}_X \cong \mathcal{K}_X$.

- This is a highbrow way of saying "rational functions are regular functions which are defined on a dense open subset".
- Another reformulation is that $\mathcal{K}_X$ is the sheafification of $\mathcal{O}_X$ with respect to the $\neg\neg$-modality.
- There is a generalization to nonreduced schemes.

Group schemes

Motto: Internal to $\mathrm{Zar}(S)$, group schemes look like ordinary groups.

group scheme	internal definition	functor of points: $X \mapsto \ldots$
$\mathbb{G}_\mathrm{a}$	$\underline{\mathbb{A}}^1$ (as additive group)	$\mathcal{O}_X(X)$
$\mathbb{G}_\mathrm{m}$	$\{x : \underline{\mathbb{A}}^1 \mid \ulcorner x \text{ inv.} \urcorner\}$	$\mathcal{O}_X(X)^\times$
μ_n	$\{x : \underline{\mathbb{A}}^1 \mid x^n = 1\}$	$\{f \in \mathcal{O}_X(X) \mid f^n = 1\}$
GL_n	$\{M : \underline{\mathbb{A}}^{1^{n \times n}} \mid \ulcorner M \text{ inv.} \urcorner\}$	$\mathrm{GL}_n(\mathcal{O}_X(X))$

Applications in algebra

Let A be a commutative ring. The internal language of $\mathrm{Sh}(\operatorname{Spec} A)$ allows you to say "without loss of generality, we may assume that A is local", even constructively.

The kernel of any matrix over a principial ideal domain is finitely generated.

$$\Downarrow$$

The kernel of any matrix over a Prüfer domain is finitely generated.

Hilbert's program in algebra

There is a way to combine some of the powerful tools of classical ring theory with the advantages that constructive reasoning provides, for instance exhibiting explicit witnesses. Namely we can devise a language in which we can usefully talk about prime ideals, but which substitutes non-constructive arguments by constructive arguments "behind the scenes". The key idea is to substitute the phrase "for all prime ideals" (or equivalently "for all prime filters") by "for the generic prime filter".

More specifically, simply interpret a given proof using prime filters in $\mathrm{Sh}(\operatorname{Spec} A)$ and let it refer to $\mathcal{F} \hookrightarrow \underline{A}$.

Statement	constructive substitution	meaning
$x \in \mathfrak{p}$ for all $\mathfrak{p}$.	$x \notin \mathcal{F}$.	x is nilpotent.
$x \in \mathfrak{p}$ for all $\mathfrak{p}$ such that $y \in \mathfrak{p}$.	$x \in \mathcal{F} \Rightarrow y \in \mathcal{F}$.	$x \in \sqrt{(y)}$.
x is regular in all stalks $A_\mathfrak{p}$.	x is regular in $\underline{A}[\mathcal{F}^{-1}]$.	x is regular in A.
The stalks $A_\mathfrak{p}$ are reduced.	$\underline{A}[\mathcal{F}^{-1}]$ is reduced.	A is reduced.
The stalks $M_\mathfrak{p}$ vanish.	$\underline{M}[\mathcal{F}^{-1}] = 0$.	$M = 0$.
The stalks $M_\mathfrak{p}$ are flat over $A_\mathfrak{p}$.	$\underline{M}[\mathcal{F}^{-1}]$ is flat over $\underline{A}[\mathcal{F}^{-1}]$.	M is flat over A.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are injective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is injective.	$M \to N$ is injective.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are surjective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is surjective.	$M \to N$ is surjective.

This is related (in a few cases equivalent) to the *dynamical methods in algebra* explored by Coquand, Coste, Lombardi, Roy, and others. Their approach is more versatile.

The curious role of affine open subsets

Question: Why do the following identities hold, for quasicoherent sheaves $\mathcal{E}$ and $\mathcal{F}$ and affine open subsets U?

$$\begin{aligned}
(\mathcal{E}/\mathcal{F})(U) &= \mathcal{E}(U)/\mathcal{F}(U) \\
(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})(U) &= \mathcal{E}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \\
\mathcal{E}_{\mathrm{tors}}(U) &= \mathcal{E}(U)_{\mathrm{tors}} \quad \text{(sometimes)} \\
\mathcal{K}_X(U) &= \operatorname{Quot} \mathcal{O}_X(U) \quad \text{(sometimes)}
\end{aligned}$$

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$$\begin{aligned}(\mathcal{E}/\mathcal{F})(U) &= \mathcal{E}(U)/\mathcal{F}(U)\\ (\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{F})(U) &= \mathcal{E}(U)\otimes_{\mathcal{O}_X(U)}\mathcal{F}(U)\\ \mathcal{E}_{\mathrm{tors}}(U) &= \mathcal{E}(U)_{\mathrm{tors}} \quad \text{(sometimes)}\\ \mathcal{K}_X(U) &= \operatorname{Quot}\mathcal{O}_X(U) \quad \text{(sometimes)}\end{aligned}$$

A calculation:

$$\begin{aligned}M^\sim\otimes_{\mathcal{O}_U}N^\sim &= \underline{M}[\mathcal{F}^{-1}]\otimes_{\underline{A}[\mathcal{F}^{-1}]}\underline{N}[\mathcal{F}^{-1}] = (\underline{M}\otimes_{\underline{A}}\underline{N})[\mathcal{F}^{-1}]\\ &= (\underline{M\otimes_A N})[\mathcal{F}^{-1}] = (M\otimes_A N)^\sim.\end{aligned}$$

The curious role of affine open subsets

Question: Why do the following identities hold, for quasicoherent sheaves $\mathcal{E}$ and $\mathcal{F}$ and affine open subsets U?

$$\begin{aligned}(\mathcal{E}/\mathcal{F})(U) &= \mathcal{E}(U)/\mathcal{F}(U) \\ (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})(U) &= \mathcal{E}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \\ \mathcal{E}_{\mathrm{tors}}(U) &= \mathcal{E}(U)_{\mathrm{tors}} \quad \text{(sometimes)} \\ \mathcal{K}_X(U) &= \operatorname{Quot} \mathcal{O}_X(U) \quad \text{(sometimes)}\end{aligned}$$

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Answer: Because localisation commutes with quotients, tensor products, torsion submodules (sometimes), …

Quasicoherence

Let X be a scheme. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module.

Then $\mathcal{E}$ is quasicoherent if and only if, internally to $\mathrm{Sh}(X)$,

$$\mathcal{E}[f^{-1}] \text{ is a } \Box_f\text{-sheaf for any } f : \mathcal{O}_X,$$
$$\text{where } \Box_f \varphi :\equiv (f \text{ invertible} \Rightarrow \varphi).$$

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In particular: If $\mathcal{E}$ is quasicoherent, then internally

$$(f \text{ invertible} \Rightarrow s = 0) \Longrightarrow \bigvee_{n \geq 0} f^n s = 0$$

for any $f : \mathcal{O}_X$ and $s : \mathcal{E}$.

The $\Box$-translation

Let $\mathcal{E}_\Box \hookrightarrow \mathcal{E}$ be a subtopos given by a local operator. Then

$$\mathcal{E}_\Box \models \varphi \qquad \text{iff} \qquad \mathcal{E} \models \varphi^\Box,$$

where the translation $\varphi \mapsto \varphi^\Box$ is given by:

$$\begin{aligned}
(s = t)^\Box &:\equiv \Box(s = t) \\
(\varphi \wedge \psi)^\Box &:\equiv \Box(\varphi^\Box \wedge \psi^\Box) \\
(\varphi \vee \psi)^\Box &:\equiv \Box(\varphi^\Box \vee \psi^\Box) \\
(\varphi \Rightarrow \psi)^\Box &:\equiv \Box(\varphi^\Box \Rightarrow \psi^\Box) \\
(\forall x \colon X.\, \varphi(x))^\Box &:\equiv \Box(\forall x \colon X.\, \varphi^\Box(x)) \\
(\exists x \colon X.\, \varphi(x))^\Box &:\equiv \Box(\exists x \colon X.\, \varphi^\Box(x))
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$$\mathcal{E}_\Box \models \varphi \qquad \text{iff} \qquad \mathcal{E} \models \varphi^\Box.$$

Let X be a scheme. Depending on $\Box$, $\mathrm{Sh}(X) \models \Box\varphi$ means that φ holds on …

- … a dense open subset.
- … a schematically dense open subset.
- … a given open subset U.
- … an open subset containing a given closed subset A.
- … an open neighbourhood of a given point $x \in X$.

Can tackle the question "$\varphi^\Box \overset{?}{\Rightarrow} \Box\varphi$" logically.

The absolute spectrum, internalised

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\operatorname{Spec} A := \text{topological space of the prime ideals of } A$$

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The frame of opens of $\operatorname{Spec} A$ is the frame of radical ideals in A.
Universal property:

$$\mathrm{Hom}_{\mathrm{LRT}/|\mathcal{E}|}(T, \operatorname{Spec} A) \cong \mathrm{Hom}_{\mathrm{Ring}(\mathcal{E})}(A, \mu_* \mathcal{O}_T)$$

for all locally ringed toposes T equipped with a geometric morphism $T \xrightarrow{\mu} \mathcal{E}$.

The relative spectrum

Let X be a scheme and $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra. Can we describe its **relative spectrum** $\operatorname{Spec}_X \mathcal{A} \to X$ internally?

Desired universal property:

$$\operatorname{Hom}_{\mathrm{LRL}/X}(T, \operatorname{Spec}_X \mathcal{A}) \cong \operatorname{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_* \mathcal{O}_T)$$

for all locally ringed locales T over X.

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Beware of believing false statements

- $\operatorname{Spec}_X \mathcal{O}_X = X$.
- $\operatorname{Spec} \mathcal{A}$ is the one-point locale iff every element of $\mathcal{A}$ is invertible or nilpotent.
- Every element of $\mathcal{O}_X$ which is not invertible is nilpotent.
- Thus cannot prove $\operatorname{Spec} \mathcal{O}_X = \mathrm{pt}$ internally.

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for all locally ringed locales T over X.

Solution: Define internally the frame of $\operatorname{Spec}_X \mathcal{A}$ to be the frame of those radical ideals $I \subseteq \mathcal{A}$ such that

$$\forall f \colon \mathcal{O}_X.\, \forall s \colon \mathcal{A}.\, (f \text{ invertible in } \mathcal{O}_X \Rightarrow s \in I) \Longrightarrow fs \in I.$$

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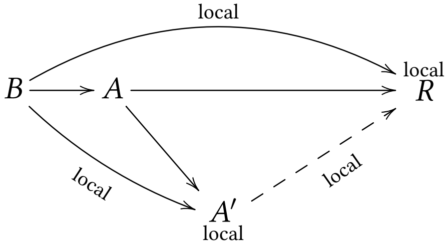
Its **points** are those prime filters G of $\mathcal{A}$ such that

$$\forall f \colon \mathcal{O}_X. \varphi(f) \in G \Longrightarrow f \text{ invertible in } \mathcal{O}_X.$$

The relative spectrum, reformulated

Let $B \to A$ be an algebra in a topos.

Is there a **free local and local-over-B ring** $A \to A'$ over A?



Form limits in the category of **locally ringed locales** by **relocalising** the corresponding limit in ringed locales.

The étale subtopos

Recall that the **Kummer sequence** is not exact in $\mathrm{Zar}(S)$ at the third term:

$$1 \longrightarrow \mu_n \longrightarrow (\underline{\mathbb{A}}^1)^\times \xrightarrow{(\cdot)^n} (\underline{\mathbb{A}}^1)^\times \longrightarrow 1$$

But we have:

$$\mathrm{Zar}(S) \models \forall f \colon (\underline{\mathbb{A}}^1)^\times.\, \Box_{\text{ét}}(\exists g : (\underline{\mathbb{A}}^1)^\times.\, f = g^n),$$

where $\Box_{\text{ét}}$ is such that $\mathrm{Zar}(S)_{\Box_{\text{ét}}} \hookrightarrow \mathrm{Zar}(S)$ is the **big étale topos** of S. It is the largest subtopos of $\mathrm{Zar}(S)$ where

$$\ulcorner \underline{\mathbb{A}}^1 \text{ is separably closed} \urcorner$$

holds [reinterpretation of Wraith, PSSL 1].