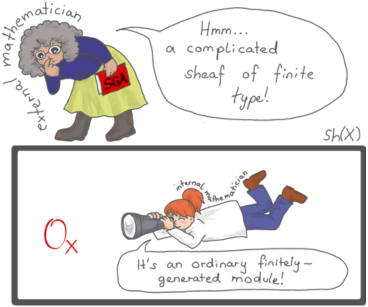

Using the internal language of toposes in algebraic geometry

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Photo source: Newtown News

Outline

Abstract

We describe how the internal language of certain toposes, the associated small and big Zariski toposes of a scheme, can be used to give simpler definitions and more conceptual proofs of the basic notions and observations in algebraic geometry.

The starting point is that, from the internal point of view, sheaves of rings and sheaves of modules look just like plain rings and plain modules. In this way, some concepts and statements of scheme theory can be reduced to concepts and statements of intuitionistic linear algebra.

Furthermore, modal operators can be used to model phrases such as "on a dense open subset it holds that" or "on an open neighbourhood of a given point it holds that". These operators define certain subtoposes; a generalization of the double-negation translation is useful in order to understand the internal universe of those subtoposes from the internal point of view of the ambient topos.

A particularly interesting task is to internalise the construction of the relative spectrum, which, given a quasicoherent sheaf of algebras on a scheme X, yields a scheme over X. From the internal point of view, this construction should simply reduce to an intuitionistically sensible variant of the ordinary construction of the spectrum of a ring, but it turns out that this expectation is too naive and that a refined approach is necessary.

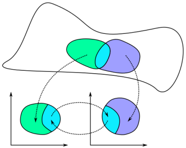
We also discuss how the little Zariski topos can be described using the internal language of the big Zariski topos, and vice versa; here too there is a small surprise.

What is a scheme?

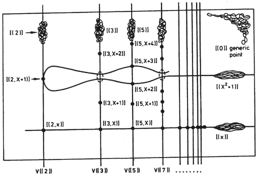
- A **manifold** is a space which is **locally isomorphic** to some open subset of some $\mathbb{R}^n$.
- A **scheme** is a space which is **locally isomorphic** to the **spectrum of some (commutative) ring**:

$$\operatorname{Spec} A := \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal}\}$$

- By **space** we mean: topological space X equipped with a local sheaf $\mathcal{O}_X$ of rings.



a manifold



Mumford's treasure map of $\operatorname{Spec} \mathbb{Z}[X]$

A *sheaf of rings* on a topological space X is a ring object in $\mathrm{Sh}(X)$, the category of set-valued sheaves on X.

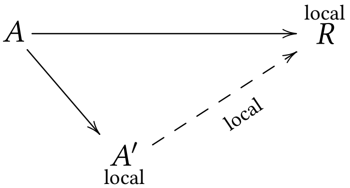
A sheaf $\mathcal{O}_X$ of rings is *local* if and only if all the stalks $\mathcal{O}_{X,x}$ are local rings. Why not demand that the sets of sections $\mathcal{O}_X(U)$ are local rings? This choice has a geometric meaning, but can also be motivated from a logical point of view: A sheaf of rings is local if and only if, from the point of view of the internal language of $\mathrm{Sh}(X)$, it is a local ring.

Think of $\mathcal{O}_X$ as the sheaf of "number-valued functions" on X. In algebraic geometry, this structure sheaf is a crucial part of the data: Wildly different schemes can have the same underlying topological space.

Motivating the spectrum

Let A be a commutative ring (in Set).

Is there a **free local ring** $A \to A'$ over A?



No, if we restrict to Set.

Yes, if we allow a change of topos: Then $A \to \mathcal{O}_{\operatorname{Spec} A}$ is the universal localization.

Details on this point of view can be found in one of Peter Arndt's very nice answers on MathOverflow:

http://mathoverflow.net/a/14334/31233

What is a topos?

Formal definition

A **topos** is a category which has finite limits, is cartesian closed and has a subobject classifier.

Motto

A topos is a category which is sufficiently rich to support an **internal language**.

Examples

- Set: category of sets
- $\mathrm{Sh}(X)$: category of set-valued sheaves on a space X

While technically correct, the formal definition is actually misleading in a sense: A topos has lots of other vital structure, which is crucial for a rounded understanding, but is not listed in the definition (which is trimmed for minimality).

A more comprehensive definition is: A *topos* is a locally cartesian closed, finitely complete and cocomplete Heyting category which is exact, extensive and has a subobject classifier.

Check out an article by Tom Leinster for a leisurely introduction to topos theory.

What is the internal language?

The internal language of a topos $\mathcal{E}$ allows to

1. construct objects and morphisms of the topos,
2. formulate statements about them and
3. prove such statements

in a **naive element-based** language:

externally	internally to $\mathcal{E}$
object of $\mathcal{E}$	set/type
morphism in $\mathcal{E}$	map of sets
monomorphism	injective map
epimorphism	surjective map
group object	group

The internal language of $\mathrm{Sh}(X)$

Let X be a topological space. Then we recursively define

$$U \models \varphi \quad \text{(“}\varphi\text{ holds on }U\text{”)}$$

for open subsets $U \subseteq X$ and formulas φ. Write “$\mathrm{Sh}(X) \models \varphi$” to mean $X \models \varphi$.

The internal language of $\mathrm{Sh}(X)$

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for open subsets $U \subseteq X$ and formulas φ. Write “$\mathrm{Sh}(X) \models \varphi$” to mean $X \models \varphi$.

$$
\begin{aligned}
U \models f = g : \mathcal{F} \quad &:\Longleftrightarrow f|_U = g|_U \in \mathcal{F}(U) \\
U \models \varphi \wedge \psi \quad &:\Longleftrightarrow U \models \varphi \text{ and } U \models \psi \\
U \models \varphi \vee \psi \quad &:\Longleftrightarrow \sout{U \models \varphi \text{ or } U \models \psi} \\
&\quad\ \text{there exists a covering } U = \textstyle\bigcup_i U_i \text{ s. th. for all } i\text{:} \\
&\qquad\quad U_i \models \varphi \text{ or } U_i \models \psi \\
U \models \varphi \Rightarrow \psi \quad &:\Longleftrightarrow \text{for all open } V \subseteq U\text{: } V \models \varphi \text{ implies } V \models \psi \\
U \models \forall f : \mathcal{F}.\, \varphi(f) &:\Longleftrightarrow \text{for all sections } f \in \mathcal{F}(V),\, V \subseteq U\text{: } V \models \varphi(f) \\
U \models \exists f : \mathcal{F}.\, \varphi(f) &:\Longleftrightarrow \text{there exists a covering } U = \textstyle\bigcup_i U_i \text{ s. th. for all } i\text{:} \\
&\qquad\quad \text{there exists } f_i \in \mathcal{F}(U_i) \text{ s. th. } U_i \models \varphi(f_i)
\end{aligned}
$$

- Special case: The language of Set is the usual mathematical language.

- Actually, the objects of $\mathcal{E}$ feel more like *types* instead of *sets*: For instance, there is no global membership relation $\in$. Rather, for each object A of $\mathcal{E}$, there is a relation $\in_A : A \times \mathcal{P}(A) \to \Omega$, where $\mathcal{P}(A)$ is the power object of A and Ω is the object of truth values of $\mathcal{E}$ (can be understood as the power object of a terminal object).

- Compare with the embedding theorem for abelian categories: There, an explicit embedding into a category of modules is constructed. Here, we only change perspective and talk about the same objects and morphisms.

- There exists a weaker variant of the internal language which works in abelian categories. By using it, one can even pretend that the objects are abelian groups (instead of modules), and when constructing morphisms by appealing to the axiom of unique choice (which is a theorem), one doesn't even have to check linearity. The proof that this approach works uses only categorical logic.

- For expositions of the internal language, see Chapters D1 to D4 of the Elephant, Chapter VI of Moerdijk and Mac Lane's book, or Chapter 13 of these lecture notes by Thomas Streicher.

- The internal language of a sheaf topos of a T_1-space is *classical* (that is, verifies the principle of excluded middle) if and only if the space is discrete. That's a not particularly interesting special case.

- See Section 2.4 of these notes for remarks on how to appreciate intuitionistic logic.

- The rules are called *Kripke–Joyal semantics* and can be formulated over any topos (not just sheaf topoi). They are not all arbitrary: Rather, they are very finely concerted to make the crucial properties about the internal language (see next slide) true.

- If $\mathcal{F}$ is an object of $\mathrm{Sh}(X)$, we write "$f : \mathcal{F}$" instead of "$f \in \mathcal{F}$" to remind us that $\mathcal{F}$ is not really (externally) a set consisting of elements, but that we only pretend this by using the internal language.

- There are two further rules concerning the constants $\top$ and $\bot$ (truth resp. falsehood):

$$\begin{aligned} U \models \top \quad &:\Longleftrightarrow \quad U = U \,\text{(always fulfiled)} \\ U \models \bot \quad &:\Longleftrightarrow \quad U = \emptyset \end{aligned}$$

- Negation is defined as $\neg\varphi :\equiv (\varphi \Rightarrow \bot)$.

- The alternate definition "$U \models \varphi \vee \psi :\Leftrightarrow U \models \varphi$ or $U \models \psi$" would not be local (cf. next slide).

- Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then:

$$
\begin{aligned}
& X \models \ulcorner \alpha \text{ is injective} \urcorner \\
\iff & X \models \forall s, t \colon \mathcal{F}.\, \alpha(s) = \alpha(t) \Rightarrow s = t \\
\iff & \text{for all open } U \subseteq X, \text{ sections } s, t \in \mathcal{F}(U): \\
& \qquad U \models \alpha(s) = \alpha(t) \Rightarrow s = t \\
\iff & \text{for all open } U \subseteq X, \text{ sections } s, t \in \mathcal{F}(U): \\
& \qquad \text{for all open } V \subseteq U: \\
& \qquad\qquad \alpha_V(s|_V) = \alpha_V(t|_V) \text{ implies } s|_V = t|_V \\
\iff & \text{for all open } U \subseteq X, \text{ sections } s, t \in \mathcal{F}(U): \\
& \qquad \alpha_U(s|_U) = \alpha_U(t|_U) \text{ implies } s|_U = t|_U \\
\iff & \alpha \text{ is a monomorphism of sheaves}
\end{aligned}
$$

- The corner quotes "$\ulcorner \ldots \urcorner$" indicate that translation into formal language is left to the reader.

- Similarly, we have (exercise, use the rules!):

$$
\begin{aligned}
& X \models \ulcorner \alpha \text{ is surjective} \urcorner \\
\iff & X \models \forall s\!:\!\mathcal{G}.\, \exists t\!:\!\mathcal{F}.\, \alpha(t) = s \\
\iff & \alpha \text{ is an epimorphism of sheaves}
\end{aligned}
$$

- One can simplify the rules for often-occuring special cases:

$$U \models \forall s : \mathcal{F}.\, \forall t : \mathcal{G}.\, \varphi(s,t) \quad \Longleftrightarrow \quad \begin{array}{l} \text{for all open } V \subseteq U, \\ \text{sections } s \in \mathcal{F}(V),\, t \in \mathcal{G}(V): \\ \qquad V \models \varphi(s,t) \end{array}$$

$$U \models \forall s : \mathcal{F}.\, \varphi(s) \Rightarrow \psi(s) \quad \Longleftrightarrow \quad \begin{array}{l} \text{for all open } V \subseteq U, \text{ sections } s \in \mathcal{F}(V): \\ \qquad V \models \varphi(s) \text{ implies } V \models \psi(s) \end{array}$$

$$U \models \exists! s : \mathcal{F}.\, \varphi(s) \quad \Longleftrightarrow \quad \begin{array}{l} \text{for all open } V \subseteq U, \\ \text{there is exactly one section } s \in \mathcal{F}(V) \text{ with:} \\ \qquad V \models \varphi(s) \end{array}$$

- One can extend the language to allow for *unbounded* quantification ($\forall A$ vs. $\forall a \in A$), by Shulman's *stack semantics*. This is needed to formulate universal properties internal to $\mathrm{Sh}(X)$, for instance.

- One can further extend the language to be able to talk about locally internal categories over $\mathrm{Sh}(X)$ (in the sense of Penon, see for instance the appendix of Johnstone's first topos theory book): Then one can do category theory internal to $\mathrm{Sh}(X)$ using the internal language.

  This specific approach is, as far as I am aware, original work. But of course, internal category theory has been done for a long time, see for instance the Elephant and also Chapman and Rowbottom's *Relative Category Theory and Geometric Morphisms: A Logical Approach*.

The internal language of $\mathrm{Sh}(X)$

Crucial property: Locality

If $U = \bigcup_i U_i$, then $U \models \varphi$ iff $U_i \models \varphi$ for all i.

Crucial property: Soundness

If $U \models \varphi$ and φ implies ψ constructively, then $U \models \psi$.

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no $\varphi \vee \neg\varphi$, no $\neg\neg\varphi \Rightarrow \varphi$, no AxC

A first glance at the constructive nature

- $U \models f = 0$ iff $f|_U = 0 \in \mathcal{F}(U)$.
- $U \models \neg\neg(f = 0)$ iff $f = 0$ on a dense open subset of U.

Why is constructive mathematics interesting?

- The internal logic of most toposes is constructive.
- From a constructive proof of a statement, it's always possible to mechanically extract an *algorithm* witnessing its truth. For example: A proof of the infinitude of primes gives rise to an algorithm which actually computes infinitely many primes (outputting one at a time, never stopping).
- By the celebrated *Curry–Howard correspondence*, constructive truth of a formula is equivalent to the existence of a program of a certain type associated to the formula.
- In constructive mathematics, one can experiment with (and draw useful conclusions also holding in a usual sense) *anti-classical dream axioms*, for instance the one of synthetic differential geometry:

  All functions $\mathbb{R} \to \mathbb{R}$ *are smooth.*

- Constructive accounts of classical theories are sometimes more elegant or point out some minor but interesting points which are not appreciated by a classical perspective.
- The philosophical question on the *meaning of truth* is easier to tackle in constructive mathematics.

Three rumours about constructive mathematics

1. There is a false rumour about constructive mathematics, namely that the term *contradiction* is generally forbidden. This is not the case, one has to distinguish between

   - a true proof by contradiction: "Assume φ were false. Then ..., contradiction. So φ is in fact true."

   which constructively is only a proof of the weaker statement $\neg\neg\varphi$, and

   - a proof of a negated formula: "Assume ψ were true. Then ..., contradiction. So $\neg\psi$ holds."

   which is a perfectly fine proof of $\neg\psi$ in constructive mathematics.

2. There is a similar rumour that constructive mathematicians *deny* the law of excluded middle. In fact, one can constructively prove that there is no counterexample to the law: For any formula φ, it holds that $\neg\neg(\varphi \vee \neg\varphi)$.

   In constructive mathematics, one merely doesn't *use* the law of excluded middle. (Only in concrete models, for example as provided by the internal universe of the sheaf topos on a non-discrete topological space, the law of excluded middle will actually be refutable.)

3. There is one last false rumour about constructive mathematics: Namely that most of mathematics breaks down in a constructive setting. This is only true if interpreted naively: Often, already very small changes to the definitions and statements (which are classically simply equivalent reformulations) suffice to make them constructively acceptable.

   In other cases, adding an additional hypothesis, which is classically always satisfied, is necessary (and interesting). Here is an example: In constructive mathematics, one can not show that any inhabited subset of the natural numbers possesses a minimal element. [One can also not show the negation – recall the previous false rumour.] But one can show (quite easily, by induction) that any inhabited and *detachable* subset of the natural numbers possesses a minimal element. A subset $U \subseteq \mathbb{N}$ is detachable iff for any number $n \in \mathbb{N}$, it holds that $n \in U$ or $n \notin U$.

   This has a computational interpretation: Given an arbitrary inhabited subset $U \subseteq \mathbb{N}$, one cannot algorithmically find its minimal element. But it *is* possible if one has an algorithmic *test of membership* for U.

References about constructive mathematics include:

- Bridges. Constructive Mathematics.
- van Dalen. Intuitionistic logic.
- Troelstra and van Dalen. Constructivism in Mathematics: An Introduction.

Andrej Bauer's blog is also very informative.

The little Zariski topos

Definition

The **little Zariski topos** of a scheme X is the category $\mathrm{Sh}(X)$ of set-valued sheaves on X.

Basic look and feel

- Internally, the structure sheaf $\mathcal{O}_X$ looks like

  an ordinary ring.

- Internally, a sheaf of $\mathcal{O}_X$-modules looks like

  an ordinary module on that ring.

Building a dictionary

Understand notions of algebraic geometry as notions of algebra internal to Sh(X).

externally	internally to Sh(X)
sheaf of sets	set/type
morphism of sheaves	map of sets
monomorphism	injective map
epimorphism	surjective map
sheaf of rings	ring
sheaf of modules	module
sheaf of finite type	finitely generated module
finite locally free sheaf	finite free module
tensor product of sheaves	tensor product of modules
sheaf of Kähler differentials	module of Kähler differentials
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$
dimension of X	Krull dimension of $\mathcal{O}_X$

Building a dictionary

Understand notions of algebraic geometry as notions of algebra internal to $\mathrm{Sh}(X)$.

externally	internally to $\mathrm{Sh}(X)$
sheaf	
morp	
monc	
epim	
sheaf	
sheaf	
sheaf	
finite	
tenso	
sheaf	als
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$
dimension of X	Krull dimension of $\mathcal{O}_X$

MISCONCEPTIONS ABOUT K_X

by Steven L. KLEIMAN

There are three common misconceptions about the sheaf K_X of meromorphic functions on a ringed space X: (1) that K_X can be defined as the sheaf associated to the presheaf of total fraction rings,

(*) $$U \mapsto \Gamma(U, O_X)_{tot},$$

see [EGA IV$_4$, 20.1.3, p. 227] and [1, (3.2), p. 137]; (2) that the stalks $K_{X,x}$ are equal to the total fraction rings $(O_{X,x})_{tot}$, see [EGA IV$_4$, 20.1.1 and 20.1.3, pp. 226-7]; and (3) that if X is a scheme and $U = \mathrm{Spec}\,(A)$ is

See the notes for more dictionary entries.

The simple definition of $\mathcal{K}_X$ allows to give an internal account of the basics of the theory of Cartier divisors, for instance giving an easy description of the line bundle associated to a Cartier divisor.

Using the dictionary

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If M' and M'' are finitely generated, so is M.

$$\Downarrow$$

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are of finite type, so is $\mathcal{F}$.

Using the dictionary

Any finitely generated vector space does *not not* possess a basis.

$$\Downarrow$$

Any sheaf of modules of finite type on a reduced scheme is locally free *on a dense open subset.*

Ravi Vakil: “Important hard exercise” (13.7.K).

The objective

Understand notions and statements of **algebraic geometry** as notions and statements of (intuitionistic) **commutative algebra** internal to suitable **toposes**.

Further topics in the little Zariski topos:

- Upper semicontinuous rank function
- Transfer principles $M \leftrightarrow M^{\sim}$
- The curious role of affine open subsets
- Quasicoherence
- Spreading from points to neighbourhoods
- The relative spectrum

Praise for Mike Shulman

[1004.3802] Stack se ×

arxiv.org/abs/1004.3802

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Mathematics > Category Theory

Stack semantics and the comparison of material and structural set theories

Michael A. Shulman

(Submitted on 21 Apr 2010)

We extend the usual internal logic of a (pre)topos to a more general interpretation, called the stack semantics, which allows for "unbounded" quantifiers ranging over the class of objects of the topos. Using well-founded relations inside the stack semantics, we can then recover a membership-based (or "material") set theory from an arbitrary topos, including even set-theoretic axiom schemas such as collection and separation which involve unbounded quantifiers. This construction reproduces the models of Fourman-Hayashi and of algebraic set theory, when the latter apply. It turns out that the axioms of collection and replacement are always valid in the stack semantics of any topos, while the axiom of separation expressed in the stack semantics gives a new topos-theoretic axiom schema with the full strength of ZF. We call a topos satisfying this schema "autological."

Comments: 64 pages
Subjects: **Category Theory (math.CT)**
MSC classes: 18B25 (Primary) 03G30 (Secondary)
Cite as: arXiv:1004.3802 [math.CT]
(or arXiv:1004.3802v1 [math.CT] for this version)

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The internal language of a topos supports

- first-order logic,
- higher-order logic (for instance quantification over subsets),
- dependent types, and
- unbounded quantification.

The first three items are standard. The fourth is due to Mike Shulman. Combined, it's possible to interpret "essentially all of constructive mathematics" internal to a topos.

Restrictions persist for operations with a "set-theoretical flavor" like building an infinite union of iterated powersets, for example $\bigcup_{n\in\mathbb{N}} P^n(\mathbb{N})$.

The rank function of sheaves of modules

There is the following one-to-one correspondence:

upper semi-continuous functions on X	$\overset{1:1}{\longleftrightarrow}$	completed natural numbers
external		internal

Let M be a f. g. A-module. Assume that A is a field. Then M is free iff the minimal number of generators is an actual natural number.

$\Longrightarrow$

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Assume that X is reduced. Then $\mathcal{F}$ is locally free iff its rank is locally constant.

Proposition

If every inhabited subset of the natural numbers has a minimum, then the law of excluded middle holds. (So in constructive mathematics, one cannot prove the natural numbers to be complete in this sense.)

Proof

Let φ be an arbitrary formula. Define the subset

$$U := \{n \in \mathbb{N} \mid n = 1 \vee \varphi\} \subseteq \mathbb{N},$$

which surely is inhabited by $1 \in U$. So by assumption, there exists a number $z \in \mathbb{N}$ which is the minimum of U. We have

$$z = 0 \quad \vee \quad z > 0$$

(this is constructively not trivial, but can be proven by induction).
If $z = 0$, we have $0 \in U$, so $0 = 1 \vee \varphi$, so φ holds.
If $z > 0$, then $\neg\varphi$ holds: Because if φ were true, zero would be an element of U, contradicting the minimality of z.

Proposition

The partially ordered set

$$\widehat{\mathbb{N}} := \{A \subseteq \mathbb{N} \mid A \text{ inhabited and upward closed}\}$$

is the least partially ordered set containing $\mathbb{N}$ and possessing minima of arbitrary inhabited subsets.

The embedding $\mathbb{N} \hookrightarrow \widehat{\mathbb{N}}$ is given by

$$n \in \mathbb{N} \longmapsto \uparrow(n) := \{m \in \mathbb{N} \mid m \geq n\}.$$

Proof

If $M \subseteq \widehat{\mathbb{N}}$ is an inhabited subset, its minimum is

$$\min M = \bigcup M \in \widehat{\mathbb{N}}.$$

The proof of the universal property is straightforward.

External translation (see Mulvey's *Intuitionistic algebra and representations of rings*)

Let X be a topological space and consider the constant sheaf N with $\Gamma(U, N) = \{f \colon U \to \mathbb{N} \mid f \text{ continuous}\}$. Internally, the sheaf N plays the role of the ordinary natural numbers. Then there is an one-to-one correspondence:

1. Let $A \hookrightarrow N$ be a subobject which is inhabited and upward closed from the internal point of view. Then

   $$x \longmapsto \inf\{n \in \mathbb{N} \mid n \in A_x\}$$

   is an upper semi-continuous function on X.

2. Let $\alpha : X \to \mathbb{N}$ be a upper semi-continuous function. Then

   $$U \subseteq X \longmapsto \{f \colon U \to \mathbb{N} \mid f \text{ continuous},\ \ f \geq \alpha \text{ on } U\}$$

   is a subobject of N which internally is inhabited and upward closed.

- Here is an explicit example of a completed natural number which is not an ordinary natural number: Let $X = \operatorname{Spec} k[X]$ and $\mathcal{F} = k[X]/(X-a)^{\sim}$. The rank of $\mathcal{F}$ is 1 at a and zero elsewhere. It corresponds to the internal completed natural number

$$z := \min\{n \in \mathbb{N} \mid \ulcorner\mathcal{F} \text{ can be generated by } n \text{ elements}\urcorner\} =$$
$$\min\{n \in \mathbb{N} \mid n \geq 1 \vee \ulcorner\text{the element } (X-a) \text{ of } \mathcal{O}_X \text{ is invertible}\urcorner\}.$$

We have the internal implications

$$\begin{aligned} \mathrm{Sh}(X) &\models \quad \ulcorner(X-a) \text{ inv.}\urcorner \Rightarrow z = 0 \\ \mathrm{Sh}(X) &\models \neg\ulcorner(X-a) \text{ inv.}\urcorner \Rightarrow z = 1, \end{aligned}$$

but we do *not* have

$$\mathrm{Sh}(X) \models \ulcorner(X-a) \text{ inv.}\urcorner \ \vee\ \neg\ulcorner(X-a) \text{ inv.}\urcorner,$$

which would imply

$$\mathrm{Sh}(X) \models z = 0 \vee z = 1,$$

i. e. the false statement that $\mathcal{F}$ is locally free (of ranks 0 resp. 1).

- Here is a constructive proof of the statement that finitely generated vector spaces, for which the minimal number of generators is an actual natural numbers, are free:

  By assumption, the minimal number $n \in \mathbb{N}$ of generators for M exists. Let $x_1, \ldots, x_n$ be a generating family of minimal length n. We want to verify that it's linearly independent, so that it constitutes a basis.

  Let $\sum_i \lambda_i x_i = 0$. If any λ_i were invertible, the shortened family $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ would also generate M. By minimality of n, this is not possible. So each λ_i is not invertible. By the field assumption on A, it follows that each λ_i is zero.

- In constructive mathematics, one can not show that every finitely generated vector space over a field admits a finite basis. (Exercise: Prove this by showing that this would imply the law of excluded middle.) This is not because the space might strangely turn out to be infinite-dimensional, but merely because one may not be able to explicitly exhibit a finite basis.

Transfer principles

Question: How do the properties of

- an A-module M in Set and
- the $\mathcal{O}_X$-module $M^\sim$ in $\mathrm{Sh}(X)$, where $X = \operatorname{Spec} A$, relate?

an important sheaf with $M^\sim(X) = M$

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Observation: $M^\sim = \underline{M}[\mathcal{F}^{-1}]$, where

- $\underline{M}$ is the constant sheaf with stalks M on X and
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Note: M and $\underline{M}$ share all first-order properties.

Answer: $M^\sim$ inherits those properties of M which are **stable under localization**.

The concept of a *prime filter* is a direct axiomatization of what you expect the complement of a prime ideal to fulfil. In classical logic, complementation gives a bijection between the prime filters and the prime ideals of a ring.

Prime filters are important in constructive mathematics because localizing them gives rise to local rings. In contrast, localizing a ring at the complement of a prime ideal doesn't usually result in a local ring.

To construct the universal localization of A, one doesn't pick a particular prime filter F to construct $A[F^{-1}]$. Instead, one picks the *generic prime filter* $\mathcal{F}$. This filter doesn't live in Set, but in $\mathrm{Sh}(\mathrm{Spec}\, A)$.

The curious role of affine open subsets

Question: Why do the following identities hold, for quasicoherent sheaves $\mathcal{E}$ and $\mathcal{F}$ and affine open subsets U?

$$\begin{aligned}
(\mathcal{E}/\mathcal{F})(U) &= \mathcal{E}(U)/\mathcal{F}(U) \\
(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})(U) &= \mathcal{E}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \\
\mathcal{E}_{\mathrm{tors}}(U) &= \mathcal{E}(U)_{\mathrm{tors}} \quad \text{(sometimes)} \\
\mathcal{K}_X(U) &= \operatorname{Quot} \mathcal{O}_X(U) \quad \text{(sometimes)}
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A calculation:

$$\begin{aligned}M^\sim \otimes_{\mathcal{O}_U} N^\sim &= \underline{M}[\mathcal{F}^{-1}] \otimes_{\underline{A}[\mathcal{F}^{-1}]} \underline{N}[\mathcal{F}^{-1}] = (\underline{M} \otimes_{\underline{A}} \underline{N})[\mathcal{F}^{-1}] \\ &= (\underline{M \otimes_A N})[\mathcal{F}^{-1}] = (M \otimes_A N)^\sim.\end{aligned}$$

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Answer: Because localization commutes with quotients, tensor products, torsion submodules (sometimes), …

A curious property of the structure sheaf

Let X be a scheme. Internally to $\mathrm{Sh}(X)$,

any non-invertible element of $\mathcal{O}_X$ is nilpotent.

For completeness, two further remarks should be added to this treatment of the spectrum. One is that in **E** the canonical map $A \to \Gamma_*(LA)$ is an isomorphism—i.e., the representation of A in the ring of "global sections" of LA is complete. The second, due to Mulvey in the case $\mathbf{E} = \mathbf{S}$, is that in Spec(**E**, A) the formula

$$\neg(x \in U(LA)) \Rightarrow \exists n(x^n = 0)$$

is valid. This is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas. In any case, calculations such as these are easier from the point of view of the Heyting algebra of radical ideals of A, and hence will be omitted here.

Miles Tierney. On the spectrum of a ringed topos. 1976.

Quasicoherence

Let X be a scheme. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module.

Then $\mathcal{E}$ is quasicoherent if and only if, internally to $\mathrm{Sh}(X)$,

$$\mathcal{E}[f^{-1}] \text{ is a } \Box_f\text{-sheaf for any } f : \mathcal{O}_X, \quad \text{where } \Box_f \varphi :\equiv (f \text{ invertible} \Rightarrow \varphi).$$

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In particular: If $\mathcal{E}$ is quasicoherent, then internally

$$(f \text{ invertible} \Rightarrow s = 0) \Longrightarrow \bigvee_{n \geq 0} f^n s = 0$$

for any $f : \mathcal{O}_X$ and $s : \mathcal{E}$.

The sheaf condition and the sheafification functor can be described purely internally. An object M is *separated* with respect to $\Box$ if and only if, from the internal point of view,

$$\forall x, y : M.\ \Box(x = y) \Rightarrow x = y.$$

It is a *sheaf* with respect to $\Box$, if furthermore

$$\forall K \subseteq M.\ \Box(\exists x : M.\ K = \{x\}) \Longrightarrow \exists x : M.\ \Box(x \in K).$$

The second condition displayed on the previous slide is equivalent to the separatedness condition. In the special case $\mathcal{E} = \mathcal{O}_X$, $s = 1$ it reduces to Mulvey's "somewhat obscure formula". We now understand this condition in its proper context.

The $\Box$-translation

Let $\mathcal{E}_\Box \hookrightarrow \mathcal{E}$ be a subtopos given by a local operator. Then

$$\mathcal{E}_\Box \models \varphi \qquad \text{iff} \qquad \mathcal{E} \models \varphi^\Box,$$

where the translation $\varphi \mapsto \varphi^\Box$ is given by:

$$\begin{aligned}
(s = t)^\Box &:\equiv \Box(s = t) \\
(\varphi \wedge \psi)^\Box &:\equiv \Box(\varphi^\Box \wedge \psi^\Box) \\
(\varphi \vee \psi)^\Box &:\equiv \Box(\varphi^\Box \vee \psi^\Box) \\
(\varphi \Rightarrow \psi)^\Box &:\equiv \Box(\varphi^\Box \Rightarrow \psi^\Box) \\
(\forall x : X.\, \varphi(x))^\Box &:\equiv \Box(\forall x : X.\, \varphi^\Box(x)) \\
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The $\Box$-translation

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$$\mathcal{E}_\Box \models \varphi \qquad \text{iff} \qquad \mathcal{E} \models \varphi^\Box.$$

Let X be a scheme. Depending on $\Box$, $\mathrm{Sh}(X) \models \Box\varphi$ means that φ holds on …

- … a dense open subset.
- … a schematically dense open subset.
- … a given open subset U.
- … an open subset containing a given closed subset A.
- … an open neighbourhood of a given point $x \in X$.

Can tackle the question "$\varphi^\Box \overset{?}{\Rightarrow} \Box\varphi$" logically.

The $\Box$-translation is a generalization of the *double negation translation*, which is well-known in logic. The double negation translation has the following curious property: A formula φ admits a classical proof if and only if the translated formula $\varphi^{\neg\neg}$ admits an intuitionistic proof.

The $\Box$-translation has been studied before (see for instance Aczel: *The Russell–Prawitz modality*, and Escardó, Oliva: *The Peirce translation and the double negation shift*), but to the best of my knowledge, this application – expressing the internal language of subtoposes in the internal language of the ambient topos – is new.

For ease of exposition, assume that X is irreducible with generic point ξ. Let $\Box :\equiv \neg\neg$.

Then $\mathrm{Sh}(X) \models \Box\varphi$ means that φ holds on a dense open subset of X, while $\mathrm{Sh}(X) \models \varphi^{\Box}$ means that φ holds at the generic point (taking stalks of all involved sheaves).

The question "does $\varphi^{\Box}$ imply $\Box\varphi$?" therefore means: Does φ spread from the generic point to a dense open subset?

For the special case of the double negation translation, a general answer to this purely logical question has long been known: This holds if φ is a *geometric formula* (doesn't contain $\Rightarrow$ and $\forall$).

Let $\mathcal{F}$ be a sheaf of modules on a locally ringed space X. Assume that the stalk $\mathcal{F}_x$ at some point $x \in X$ vanishes. Then in general it does *not* follow that $\mathcal{F}$ vanishes on some open neighbourhood of x.

This can be understood in logical terms: The statement that $\mathcal{F}$ vanishes,

$$\forall s : \mathcal{F}.\ s = 0,$$

is not a geometric formula.

However, if $\mathcal{F}$ is additionally supposed to be of finite type, then it *does* follow that $\mathcal{F}$ vanishes on an open neighbourhood. This too can be understood in logical terms: If $\mathcal{F}$ is of finite type, then internally there are generators $s_1, \ldots, s_n$ of $\mathcal{F}$. Thus the vanishing of $\mathcal{F}$ can be reformulated as

$$s_1 = 0 \wedge \cdots \wedge s_n = 0,$$

and this condition is manifestly geometric.

The absolute spectrum, internalised

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\operatorname{Spec} A := \text{topological space of the prime ideals of } A$$

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The frame of opens of $\operatorname{Spec} A$ is the frame of radical ideals in A. Universal property:

$$\operatorname{Hom}_{\mathrm{LRT}/|\mathcal{E}|}(T, \operatorname{Spec} A) \cong \operatorname{Hom}_{\mathrm{Ring}(\mathcal{E})}(A, \mu_* \mathcal{O}_T)$$

for all locally ringed toposes T equipped with a geometric morphism $T \xrightarrow{\mu} \mathcal{E}$.

The axioms of a prime filter constitute a propositional geometric theory. Therefore there exists the *classifying locale* over prime filters. This is the ring's spectrum. See Vicker's Locales and Toposes as Spaces and Continuity and geometric logic for very accessible introductions to this topic.

Monique Hakim constructed in her thesis a very general spectrum functor, taking a ringed topos to a locally ringed one, using explicit calculations with sites.

Using the internal language allows to reduce these calculations to a minimum. One constructs the spectrum as the sheaf topos over an internal locale and then uses the general theorem that toposes over the base $\mathcal{E}$ are the same as toposes internal to $\mathcal{E}$.

As a byproduct one obtains that Hakim's spectrum is *localic* over the base.

The relative spectrum

Let X be a scheme and $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra. Can we describe its **relative spectrum** $\underline{\mathrm{Spec}}_X\,\mathcal{A} \to X$ internally?

Desired universal property:

$$\mathrm{Hom}_{\mathrm{LRL}/X}(T, \underline{\mathrm{Spec}}_X\,\mathcal{A}) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_*\mathcal{O}_T)$$

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Beware of believing false statements

- $\underline{\mathrm{Spec}}_X\,\mathcal{O}_X = X$.
- $\mathrm{Spec}\,\mathcal{A}$ is the one-point locale iff every element of $\mathcal{A}$ is invertible or nilpotent.
- Every element of $\mathcal{O}_X$ which is not invertible is nilpotent.
- Thus cannot prove $\mathrm{Spec}\,\mathcal{O}_X = \mathrm{pt}$ internally.

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for all locally ringed locales T over X.

Solution: Define internally the frame of $\underline{\mathrm{Spec}}_X\,\mathcal{A}$ to be the frame of those radical ideals $I \subseteq \mathcal{A}$ such that

$$\forall f\colon \mathcal{O}_X.\ \forall s : \mathcal{A}.\ (f \text{ invertible in } \mathcal{O}_X \Rightarrow s \in I) \Longrightarrow fs \in I.$$

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Its **points** are those prime filters G of $\mathcal{A}$ such that

$$\forall f \colon \mathcal{O}_X. \varphi(f) \in G \Longrightarrow f \text{ invertible in } \mathcal{O}_X.$$

The stated condition on I is, under the assumption that $\mathcal{A}$ is quasicoherent, equivalent to the condition that I is quasicoherent (as an $\mathcal{O}_X$-module).

The relative spectrum is thus constructed as a certain sublocale of the absolute one. The two constructions coincide if and only if the dimension of the base scheme is ≤ 0.

If X is not a scheme or $\mathcal{A}$ is not quasicoherent, the construction still gives rise to a locally ringed locale over X which satisfies the universal property

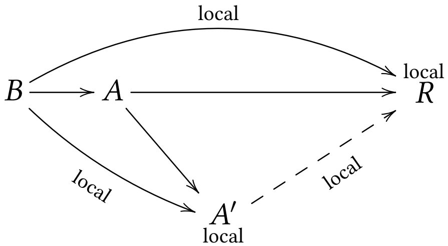
$$\mathrm{Hom}_{\mathrm{LRL}/X}(T, \underline{\mathrm{Spec}}_X \mathcal{A}) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_* \mathcal{O}_T)$$

for all locally ringed locales $T \xrightarrow{\mu} X$ over X.

The relative spectrum, reformulated

Let $B \to A$ be an algebra in a topos.

Is there a **free local and local-over-*B* ring** $A \to A'$ over A?



Form limits in the category of **locally ringed locales** by **relocalizing** the corresponding limit in ringed locales.

One might wonder whether the absolute spectrum or the relative one is “more fundamental”. The absolute spectrum can be expressed using the relative one, since

$$\operatorname{Spec} A = \underline{\operatorname{Spec}}_{\operatorname{Spec} \mathbb{Z}} A^{\sim},$$

but the other way is not in general possible: The absolute spectrum is always (quasi-)compact, while the relative one is not in general.

The big Zariski topos

Definition

The **big Zariski topos** $\mathrm{Zar}(S)$ of a scheme S is the category $\mathrm{Sh}(\mathrm{Sch}/S)$. It consists of certain functors $(\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$.

Basic look and feel

- For an S-scheme X, its functor of points

  $$\underline{X} = \mathrm{Hom}_S(\cdot, X)$$

  is an object of $\mathrm{Zar}(S)$. It feels like **the set of points** of X.
- Internally, $\underline{\mathbb{A}}^1_S$ (given by $\underline{\mathbb{A}}^1_S(X) = \mathcal{O}_X(X)$) looks like a field:

  $$\mathrm{Zar}(S) \models \forall x : \underline{\mathbb{A}}^1_S.\, x \neq 0 \Longrightarrow \ulcorner x \text{ invertible} \urcorner$$

- The overcategory Sch/S becomes a Grothendieck site by declaring families of jointly surjective open immersions to be covers. See for instance the excellent Stacks project for details.

- Working in $\mathrm{Zar}(S)$ amounts to incorporating the philosophy of describing schemes by their functors of points into one's mathematical language.

- Explicitly, the functor $\underline{X}$ is given by $\underline{X}(T) = \mathrm{Hom}_S(T, X)$ for S-schemes T. Because the Zariski site is *subcanonical*, this functor is always a sheaf.

- The object $\underline{S}$ looks like an one-element set from the internal universe. This is to be expected.

- Hakim worked out a theory of schemes internal to topoi (but without using the internal language) in her PhD thesis.

- The internal language of $\mathrm{Zar}(\operatorname{Spec} A)$ is related to the programme about dynamical methods in algebra by Coquand, Coste, Lombardi, Roy, and others. See Coquand's *A completeness proof for geometrical logic*, Coquand and Lombardi's *A logical approach to abstract algebra*, and Coste, Lombardi, and Roy's *Dynamical methods in algebra: effective Nullstellensätze*.

- The observation that $\underline{\mathbb{A}}^1_S$ is internally a field is due to Kock (in the case $S = \operatorname{Spec} \mathbb{Z}$, see his *Universal projective geometry via topos theory*) and implies a curious meta-theorem:

  Because $\mathrm{Zar}(\operatorname{Spec} \mathbb{Z})$ is the *classifying topos* for the theory of local rings, any statement about local rings which is of a certain logical form holds for the *universal model* $\underline{\mathbb{A}}^1_{\operatorname{Spec} \mathbb{Z}}$ iff it holds for any local ring (in any universe, particularly Set).

  Therefore, in proving a statement of such a form about arbitrary local rings, one may assume that they even fulfil the field condition.

  There is a similar story for local A-algebras. See Wraith's *Intuitionistic algebra: some recent developments in topos theory* for a short exposition on the usefulness of classifying topoi and universal models.

Some internal constructions

- The functor of points of $\mathbb{P}^n_S$ has the internal description

$$\{(x_0, \ldots, x_n) : (\underline{\mathbb{A}}^1_S)^{n+1} \mid x_0 \neq 0 \vee \cdots \vee x_n \neq 0\}/\text{scaling}.$$

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- Let $\mathcal{A}$ be an $\mathcal{O}_S$-algebra. This induces an $\underline{\mathbb{A}}^1_S$-algebra $\mathcal{A}^\sim$ internal to $\mathrm{Zar}(S)$. The functor of points of $\underline{\mathrm{Spec}}_S\,\mathcal{A}$ has the internal description

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$$\mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}(\mathcal{A}^\sim, \underline{\mathbb{A}}^1_S).$$

- Let X be an S-scheme. The functor of points of its tangent bundle has the internal description

$$\mathrm{Hom}(\Delta, \underline{X}),$$

where $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1_S \mid \varepsilon^2 = 0\}$.

I’m grateful to Zhen Lin Low for suggesting the example about the projective space.

Explicitly, the $\underline{\mathbb{A}}^1_S$-algebra $\mathcal{A}^\sim$ is given by

$$\mathcal{A}^\sim(X \xrightarrow{\mu} S) = (\mu^* \mathcal{A})(X).$$

A strong Kock–Lawvere axiom

- The affine line fulfils the axiom

$$\mathrm{Zar}(S) \models \ulcorner\text{every function } \underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S \text{ is a polynomial}\urcorner.$$

More precisely, the canonical morphism

$$\underline{\mathbb{A}}^1_S[T] \longrightarrow \mathrm{Hom}(\mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}(\underline{\mathbb{A}}^1_S[T], \underline{\mathbb{A}}^1_S), \underline{\mathbb{A}}^1_S)$$

is an isomorphism.

- More generally, for any $\underline{\mathbb{A}}^1_S$-algebra $\mathcal{A}$ induced by a quasicoherent $\mathcal{O}_S$-algebra, the canonical morphism

$$\mathcal{A} \longrightarrow \mathrm{Hom}(\mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}(\mathcal{A}, \underline{\mathbb{A}}^1_S), \underline{\mathbb{A}}^1_S)$$

is an isomorphism.

The étale subtopos

Recall that the **Kummer sequence** is not exact in $\mathrm{Zar}(S)$ at the third term:

$$1 \longrightarrow \mu_n \longrightarrow (\underline{\mathbb{A}}^1_S)^\times \xrightarrow{(\cdot)^n} (\underline{\mathbb{A}}^1_S)^\times \longrightarrow 1$$

But we have:

$$\mathrm{Zar}(S) \models \forall f \colon (\underline{\mathbb{A}}^1_S)^\times .\, \Box_{\text{ét}}(\exists g : (\underline{\mathbb{A}}^1_S)^\times .\, f = g^n),$$

where $\Box_{\text{ét}}$ is such that $\mathrm{Zar}(S)_{\Box_{\text{ét}}} \hookrightarrow \mathrm{Zar}(S)$ is the **big étale topos** of S. It is the largest subtopos of $\mathrm{Zar}(S)$ where

$$\ulcorner \underline{\mathbb{A}}^1_S \text{ is separably closed} \urcorner$$

holds [reinterpretation of Wraith, PSSL 1].

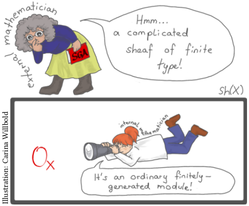
Comparing the little and the big toposes

- There is a local geometric morphism $\mathrm{Zar}(S) \to \mathrm{Sh}(S)$.
- From the point of view of $\mathrm{Sh}(S)$, the big Zariski topos is $\mathrm{Zar}(\mathcal{O}_S|\mathcal{O}_S)$, the classifying topos of local $\mathcal{O}_S$-algebras which are local over $\mathcal{O}_S$.
- From the point of view of $\mathrm{Zar}(S)$, the little Zariski topos is the largest subtopos where $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ is bijective.

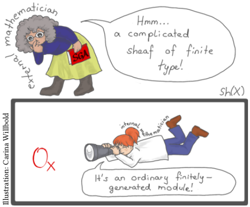
$$(\flat\underline{\mathbb{A}}^1_S)(X \xrightarrow{\mu} S) = (\mu^{-1}\mathcal{O}_S)(X)$$
$$\underline{\mathbb{A}}^1_S(X \xrightarrow{\mu} S) = \mathcal{O}_X(X)$$

Semi-open and open tasks

- Characterise quasicoherence in the big Zariski topos.
- Understand how to work with $\flat \dashv \sharp$.
- Do cohomology in the little Zariski topos; exploit that higher direct images look like ordinary sheaf cohomology from the internal point of view.
- Do cohomology in the big Zariski topos.
- Understand more subtoposes of the big Zariski topos.
- Derive suitable axioms for synthetic algebraic geometry.



Understand notions and statements of algebraic geometry as notions and statements of algebra internal to appropriate toposes.



- Simplify proofs and gain conceptual understanding.
- Understand relative geometry as absolute geometry.
- Develop a synthetic account of scheme theory.
- Contribute to constructive algebra.

http://tiny.cc/topos-notes

Participants of Augsburg's maths camp

The sun as seen from our high-altitude balloon

Translating internal statements I

Let X be a topological space (or locale) and let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then:

$$\begin{aligned}
&\mathrm{Sh}(X) \models \ulcorner \alpha \text{ is surjective} \urcorner \\
\iff &\mathrm{Sh}(X) \models \forall t : \mathcal{G}.\, \exists s : \mathcal{F}.\, \alpha(s) = t \\
\iff &\text{for all open } U \subseteq X, \text{ sections } t \in \mathcal{G}(U): \\
&\qquad \text{there exists an open covering } U = \textstyle\bigcup_i U_i \text{ and} \\
&\qquad \text{sections } s_i \in \mathcal{F}(U_i) \text{ such that:} \\
&\qquad\qquad \alpha_{U_i}(s_i) = t|_{U_i} \\
\iff &\alpha \text{ is an epimorphism of sheaves}
\end{aligned}$$

Translating internal statements II

Let X be a topological space (or locale) and let $s, t \in \mathcal{F}(X)$ be global sections of a sheaf $\mathcal{F}$ on X. Then:

$$\mathrm{Sh}(X) \models \neg\neg(s = t)$$

$$\iff \mathrm{Sh}(X) \models ((s = t) \Rightarrow \bot) \Rightarrow \bot$$

$\iff$ for all open $U \subseteq X$ such that

for all open $V \subseteq U$ such that

$s|_V = t|_V$,

it holds that $V = \emptyset$,

it holds that $U = \emptyset$

$\iff$ there exists a dense open set $W \subseteq X$ such that $s|_W = t|_W$

Spreading from points to neighbourhoods

All of the following lemmas have a short, sometimes trivial proof. Let $\mathcal{F}$ be a sheaf of finite type on a ringed space X. Let $x \in X$. Let $A \subseteq X$ be a closed subset. Then:

1. $\mathcal{F}_x = 0$ iff $\mathcal{F}|_U = 0$ for some open neighbourhood of x.
2. $\mathcal{F}|_A = 0$ iff $\mathcal{F}|_U = 0$ for some open set containing A.
3. $\mathcal{F}_x$ can be generated by n elements iff this is true on some open neighbourhood of x.
4. $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ if $\mathcal{F}$ is of finite presentation around x.
5. $\mathcal{F}$ is torsion iff $\mathcal{F}_\xi$ vanishes (assume X integral and $\mathcal{F}$ quasicoherent).
6. $\mathcal{F}$ is torsion iff $\mathcal{F}|_{\mathrm{Ass}(\mathcal{O}_X)}$ vanishes (assume X locally Noetherian and $\mathcal{F}$ quasicoherent).

Statements 1 and 2 follow from *one* proof in the internal language, applied to two different modal operators.

Similarly with statements 5 and 6.

The smallest dense sublocale

Let X be a reduced scheme satisfying a technical condition. Let $i : X_{\neg\neg} \to X$ be the inclusion of the smallest dense sublocale of X.

Then $i_* i^{-1} \mathcal{O}_X \cong \mathcal{K}_X$.

- This is a highbrow way of saying "rational functions are regular functions which are defined on a dense open subset".
- Another reformulation is that $\mathcal{K}_X$ is the sheafification of $\mathcal{O}_X$ with respect to the $\neg\neg$-modality.
- There is a generalization to nonreduced schemes.

Group schemes

Motto: Internal to $\mathrm{Zar}(S)$, group schemes look like ordinary groups.

group scheme	internal definition	functor of points: $X \mapsto \ldots$
$\mathbb{G}_\mathrm{a}$	$\underline{\mathbb{A}}^1_S$ (as additive group)	$\mathcal{O}_X(X)$
$\mathbb{G}_\mathrm{m}$	$\{x : \underline{\mathbb{A}}^1_S \mid \ulcorner x \text{ inv.} \urcorner\}$	$\mathcal{O}_X(X)^\times$
μ_n	$\{x : \underline{\mathbb{A}}^1_S \mid x^n = 1\}$	$\{f \in \mathcal{O}_X(X) \mid f^n = 1\}$
GL_n	$\{M : \underline{\mathbb{A}}^{1\ n \times n}_S \mid \ulcorner M \text{ inv.} \urcorner\}$	$\mathrm{GL}_n(\mathcal{O}_X(X))$

Applications in algebra

Let A be a commutative ring. The internal language of $\mathrm{Sh}(\mathrm{Spec}\,A)$ allows you to say "without loss of generality, we may assume that A is local", even constructively.

The kernel of any matrix over a principial ideal domain is finitely generated.

$$\Downarrow$$

The kernel of any matrix over a Prüfer domain is finitely generated.

Hilbert's program in algebra

There is a way to combine some of the powerful tools of classical ring theory with the advantages that constructive reasoning provides, for instance exhibiting explicit witnesses. Namely we can devise a language in which we can usefully talk about prime ideals, but which substitutes non-constructive arguments by constructive arguments "behind the scenes". The key idea is to substitute the phrase "for all prime ideals" (or equivalently "for all prime filters") by "for the generic prime filter".

More specifically, simply interpret a given proof using prime filters in $\mathrm{Sh}(\operatorname{Spec} A)$ and let it refer to $\mathcal{F} \hookrightarrow \underline{A}$.

Statement	constructive substitution	meaning
$x \in \mathfrak{p}$ for all $\mathfrak{p}$.	$x \notin \mathcal{F}$.	x is nilpotent.
$x \in \mathfrak{p}$ for all $\mathfrak{p}$ such that $y \in \mathfrak{p}$.	$x \in \mathcal{F} \Rightarrow y \in \mathcal{F}$.	$x \in \sqrt{(y)}$.
x is regular in all stalks $A_\mathfrak{p}$.	x is regular in $\underline{A}[\mathcal{F}^{-1}]$.	x is regular in A.
The stalks $A_\mathfrak{p}$ are reduced.	$\underline{A}[\mathcal{F}^{-1}]$ is reduced.	A is reduced.
The stalks $M_\mathfrak{p}$ vanish.	$\underline{M}[\mathcal{F}^{-1}] = 0$.	$M = 0$.
The stalks $M_\mathfrak{p}$ are flat over $A_\mathfrak{p}$.	$\underline{M}[\mathcal{F}^{-1}]$ is flat over $\underline{A}[\mathcal{F}^{-1}]$.	M is flat over A.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are injective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is injective.	$M \to N$ is injective.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are surjective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is surjective.	$M \to N$ is surjective.

This is related (in a few cases equivalent) to the *dynamical methods in algebra* explored by Coquand, Coste, Lombardi, Roy, and others. Their approach is more versatile.