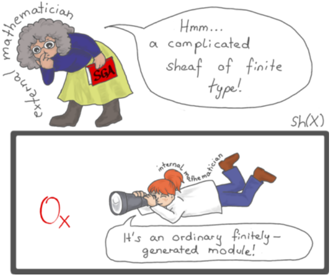

New reduction techniques in commutative algebra driven by logical methods

– interruptions welcome at any point –

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University of Verona

Colloquium Logicum 2018 in Bayreuth
September 15th, 2018

Summary

A baby example

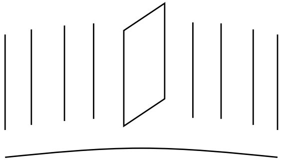
Let M be an injective matrix with more columns than rows over a ring A. Then $1 = 0$ in A.

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Generic freeness

Generically, any finitely generated module over a reduced ring is free.

(A ring is reduced iff $x^n = 0$ implies $x = 0$.)



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Let M be an injective matrix with more columns than rows over a ring A. Then $1 = 0$ in A.

Proof. Assume not. Then there is a minimal prime ideal $\mathfrak{p} \subseteq A$. The matrix is injective over the field $A_\mathfrak{p} = A[(A \setminus \mathfrak{p})^{-1}]$; contradiction to basic linear algebra.

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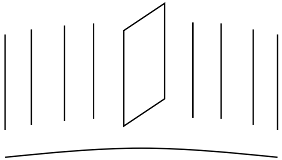

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Proof. See [Stacks Project].

Summary

- For any reduced ring A, there is a ring $A^\sim$ in a certain topos with
$$\models \left(\forall x : A^\sim .\, \neg(\exists y : A^\sim .\, xy = 1) \Rightarrow x = 0\right).$$
- This semantics is sound with respect to intuitionistic logic.
- It has uses in classical and constructive commutative algebra.

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The Kripke–Joyal semantics

Recall $A[f^{-1}] = \left\{ \frac{u}{f^n} \mid u \in A, n \in \mathbb{N} \right\}$. Let "$\models \varphi$" be short for "$1 \models \varphi$".

$$
\begin{array}{lll}
f \models \top & \text{iff} & \top \\
f \models \bot & \text{iff} & f \text{ is nilpotent} \\
f \models x = y & \text{iff} & x = y \in A[f^{-1}] \\
f \models \varphi \wedge \psi & \text{iff} & f \models \varphi \text{ and } f \models \psi \\
f \models \varphi \vee \psi & \text{iff} & \text{there exists a partition } f^n = fg_1 + \cdots + fg_m \text{ with,} \\
 & & \qquad \text{for each } i,\ fg_i \models \varphi \text{ or } fg_i \models \psi \\
f \models \varphi \Rightarrow \psi & \text{iff} & \text{for all } g \in A,\ fg \models \varphi \text{ implies } fg \models \psi \\
f \models \forall x : A^{\sim}.\, \varphi & \text{iff} & \text{for all } g \in A \text{ and all } x_0 \in A[(fg)^{-1}],\ fg \models \varphi[x_0/x] \\
f \models \exists x : A^{\sim}.\, \varphi & \text{iff} & \text{there exists a partition } f^n = fg_1 + \cdots + fg_m \text{ with,} \\
 & & \qquad \text{for each } i,\ fg_i \models \varphi[x_0/x] \text{ for some } x_0 \in A[(fg_i)^{-1}]
\end{array}
$$

The little Zariski topos of a ring

Let A be a reduced commutative ring ($x^n = 0 \Rightarrow x = 0$).

The **little Zariski topos** of A is equivalently

- the topos of sheaves over $\mathrm{Spec}(A)$,
- the locale given by the frame of radical ideals of A,
- the classifying topos of local localizations of A or
- the classifying topos of prime filters of A

and contains a **mirror image** of A, the sheaf of rings $A^\sim$.

Assuming the Boolean prime ideal theorem, a first-order formula "$\forall \ldots \forall. (\cdots \implies \cdots)$", where the two subformulas may not contain "$\Rightarrow$" and "$\forall$", holds for $A^\sim$ iff it holds for all stalks $A_\mathfrak{p}$.

$A^\sim$ inherits any property of A which is **localization-stable**.

$A^\sim$ is a **local ring** and a **field**.
$A^\sim$ has **$\neg\neg$-stable equality**.
$A^\sim$ is **anonymously Noetherian**.

The little Zariski topos of a ring

Let A be a reduced commutative ring ($x^n = 0 \Rightarrow x = 0$).

ON THE SPECTRUM OF A RINGED TOPOS 209

For completeness, two further remarks should be added to this treatment of the spectrum. One is that in **E** the canonical map $A \to \Gamma_*(LA)$ is an isomorphism—i.e., the representation of A in the ring of "global sections" of LA is complete. The second, due to Mulvey in the case **E** = **S**, is that in Spec(**E**, A) the formula

$$\neg(x \in U(LA)) \Rightarrow \exists n(x^n = 0)$$

is valid. This is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas. In any case, calculations such as these are easier from the point of view of the Heyting algebra of radical ideals of A, and hence will be omitted here.

Miles Tierney. On the spectrum of a ringed topos. 1976.

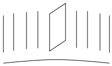
it holds for all stalks $A_\mathfrak{p}$.

$A^\sim$ is **anonymously Noetherian**.

Revisiting the test cases

Let A be a reduced commutative ring ($x^n = 0 \Rightarrow x = 0$).
Let $A^\sim$ be its mirror image in the little Zariski topos.

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



A baby example

Let M be an injective matrix over A with more columns than rows. Then $1 = 0$ in A.

Proof. M is also injective as a matrix over $A^\sim$. Since $A^\sim$ is a field, this is a contradiction by basic linear algebra. Thus $\models \bot$. This amounts to $1 = 0$ in A.

Generic freeness

Let M be a finitely generated A-module. If $f = 0$ is the only element of A such that $M[f^{-1}]$ is a free $A[f^{-1}]$-module, then $1 = 0$ in A.

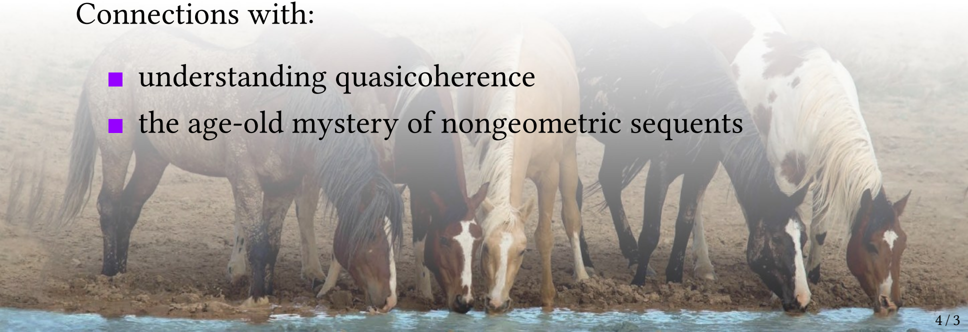
Proof. The claim amounts to $\models$ "$M^\sim$ is **not not** free". Since $A^\sim$ is a field, this follows from basic linear algebra.

The Zariski topos and related toposes have applications in:

- classical algebra and classical algebraic geometry
- constructive algebra and constructive algebraic geometry
- synthetic algebraic geometry (“schemes are just sets”)

Connections with:

- understanding quasicoherence
- the age-old mystery of nongeometric sequents



Further reading

Spiel und Spaß mit der internen Welt des kleinen Zariski-Topos

Ingo Blechschmidt

19. Dezember 2013

$R \models x = y : \mathcal{O}$	$:\Longleftrightarrow$	Für die gegebenen Elemente $x, y \in R$ gilt $x = y$.
$R \models \top$	$:\Longleftrightarrow$	$1 = 1 \in R$. (Das ist stets erfüllt.)
$R \models \bot$	$:\Longleftrightarrow$	$1 = 0 \in R$. (Das ist genau in Nullringen erfüllt.)
$R \models \phi \wedge \psi$	$:\Longleftrightarrow$	$R \models \phi$ und $R \models \psi$.
$R \models \phi \vee \psi$	$:\Longleftrightarrow$	$\cancel{R \models \phi \text{ oder } R \models \psi}$.
$R \models \phi \vee \psi$	$:\Longleftrightarrow$	Es gibt eine Zerlegung $\sum_i s_i = 1 \in R$ sodass für alle i jeweils $R[s_i^{-1}] \models \phi$ oder $R[s_i^{-1}] \models \psi$.
$R \models \phi \Rightarrow \psi$	$:\Longleftrightarrow$	Für jedes $s \in R$ gilt: Aus $R[s^{-1}] \models \phi$ folgt $R[s^{-1}] \models \psi$.
$R \models \forall x : \mathcal{O}.\ \phi$	$:\Longleftrightarrow$	Für jedes $s \in R$ und jedes $x \in R[s^{-1}]$ gilt: $R[s^{-1}] \models \phi(x)$.
$R \models \exists x : \mathcal{O}.\ \phi$	$:\Longleftrightarrow$	Es gibt eine Zerlegung $\sum_i s_i = 1 \in R$ und Elemente $x_i \in R[s_i^{-1}]$ sodass für alle i: $R[s_i^{-1}] \models \phi(x_i)$.

Die Kripke–Joyal-Semantik des kleinen Zariski-Topos.

Using the internal language of toposes in algebraic geometry

Dissertation
zur Erlangung des akademischen Grades

Dr. rer. nat.

eingereicht an der

Mathematisch-Naturwissenschaftlich-Technischen Fakultät
der Universität Augsburg

von

Ingo Blechschmidt

Universität Augsburg University

Juni 2017

Applications in algebraic geometry

Understand notions of algebraic geometry over a scheme X as notions of algebra internal to $\mathrm{Sh}(X)$.

externally	internally to $\mathrm{Sh}(X)$
sheaf of sets	set
sheaf of modules	module
sheaf of finite type	finitely generated module
tensor product of sheaves	tensor product of modules
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$
dimension of X	Krull dimension of $\mathcal{O}_X$
spectrum of a sheaf of $\mathcal{O}_X$-algebras	ordinary spectrum [with a twist]
higher direct images	sheaf cohomology

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are of finite type, so is $\mathcal{F}$.

$\Longleftarrow$

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If M' and M'' are finitely generated, so is M.

Synthetic algebraic geometry

Usual approach to algebraic geometry: **layer schemes above ordinary set theory** using either

- locally ringed spaces

  set of prime ideals of $\mathbb{Z}[X,Y,Z]/(X^n+Y^n-Z^n)$ +
  Zariski topology + structure sheaf

- or Grothendieck's functor-of-points account, where a scheme is a functor Ring $\to$ Set.

$$A \longmapsto \{(x,y,z) \in A^3 \mid x^n+y^n-z^n=0\}$$

Synthetic approach: model schemes **directly as sets** in a certain nonclassical set theory, the internal universe of the **big Zariski topos** of a base scheme.

$$\{(x,y,z) : (\underline{\mathbb{A}}^1)^3 \mid x^n+y^n-z^n=0\}$$