A GENERAL NULLSTELLENSATZ FOR GENERALIZED SPACES

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Abstract. We give a general Nullstellensatz for the generic model of a geometric theory, useful as a source of nongeometric sequents validated by the generic model, and characterize the first-order and higher-order formulas validated by the generic model.

– rough draft –

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1. Introduction

Generic models. Let $\mathbb{T}$ be a geometric theory, such as the theory of rings, the theory of local rings or the theory of intervals. We follow Caramello's terminology [12] to mean by *geometric theory* a system given by a set of sorts, a set of finitary function symbols, a set of finitary relation symbols and a set of axioms consisting of geometric sequents (sequents of the form $\varphi \vdash_{\vec{x}} \psi$ where φ and ψ are geometric formulas, that is formulas built from equality and the relation symbols by the logical connectives $\top \perp \wedge \vee \exists$ and by arbitrary set-indexed disjunctions $\bigvee$). By (infinitary) *first-order formula* we will mean a formula which may contain, additionally to the connectives allowed for geometric formulas, the connectives $\Rightarrow$ and $\forall$.

A fundamental result is that there is a *generic model* $U_{\mathbb{T}}$ of $\mathbb{T}$. This model is *conservative* in that for any geometric sequent σ, the following notions coincide:

(1) The sequent σ is provable modulo $\mathbb{T}$.
(2) The sequent σ holds for any $\mathbb{T}$-model in any Grothendieck topos.
(3) The sequent σ holds for $U_{\mathbb{T}}$.

One could argue that it is this model which we implicitly refer to when we utter the phrase "Let M be a $\mathbb{T}$-model.".[1] It can typically not be realized as a set-theoretic model, consisting of a set for each sort, a function for each function symbol and so on; instead it is a model in a custom-tailored syntactically constructed Grothendieck topos, the *classifying topos* $\mathrm{Set}[\mathbb{T}]$ of $\mathbb{T}$, hence consists of an object of $\mathrm{Set}[\mathbb{T}]$ for each sort, a morphism for each function symbol and so on.

To state what it means for a $\mathbb{T}$-structure in a topos $\mathcal{E}$ to verify the axioms of $\mathbb{T}$, rendering it a model, the *internal language* of $\mathcal{E}$ is used, roughly reviewed in Section 2.1 below. We write "$\mathcal{E} \models \alpha$" to mean that a formula α holds from the internal point of view of $\mathcal{E}$. Since this language is a form of a higher-order intuitionistic extensional dependent type theory, the classifying topos $\mathrm{Set}[\mathbb{T}]$ can be regarded as a higher-order completion of the geometric theory $\mathbb{T}$. The generic model enjoys the universal property that any $\mathbb{T}$-model in any (Grothendieck) topos $\mathcal{E}$ is the pullback of $U_{\mathbb{T}}$ along an essentially unique geometric morphism $\mathcal{E} \to \mathrm{Set}[\mathbb{T}]$.

[1] For instance, this point of view is fundamental to the slogan *continuity is geometricity*, as stressed by Vickers [22].

Nongeometric sequents. Crucially, the equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) relating provability and truth in $\mathrm{Set}[\mathbb{T}]$ only pertains to geometric sequents. The generic model may validate additional nongeometric sequents which are not provable from the axioms of $\mathbb{T}$ in first-order or even higher-order logic, and these nongeometric sequents may be quite surprising and have useful consequences.

One of the most celebrated such sequents arises in the case that $\mathbb{T}$ is the theory of local rings. In this case, the classifying topos $\mathrm{Set}[\mathbb{T}]$ is also known as the *big Zariski topos* of $\mathrm{Spec}(\mathbb{Z})$ from algebraic geometry, the topos of sheaves over the site of schemes locally of finite presentation, and the generic model is the functor $\underline{\mathbb{A}}^1$ of points of the affine line, the functor which maps any scheme X (l.o.f.p.) to $\mathrm{Hom}_{\mathrm{Sch}}(X, \mathbb{A}^1) = \mathcal{O}_X(X)$.

From the point of view of the Zariski topos, the ring object $\underline{\mathbb{A}}^1$ is not only a local ring, but even a field in the sense that any nonzero element is invertible. As this condition is of nongeometric form, it is not inherited by arbitrary local rings, which are indeed typically not fields. However, any intuitionistic consequence of this condition which is of geometric form is inherited by any local ring in any topos. Hence we may, when verifying a general fact about local rings which is expressible as a geometric sequent, suppose without loss of generality that the field axioms holds. This observation is due to Kock [16], who exploited it to develop projective geometry over local rings, and was further used by Reyes to prove a Jacobian criterion for étale morphisms [19].

We surmise that many more reduction techniques along these lines exist for other kinds of algebraic objects. However, when actually using such techniques in practice, we face the challenge that while we can use them to prove results about *all* local rings, *all* modules and so on, it is difficult to incorporate specific information about a particular local ring or a particular module at hand. This difficulty is compounded by the fact that interesting nongeometric properties are typically not inherited by the generic model of quotient theories – for instance the generic ring validates the formula $\forall x : U_{\mathbb{T}}.\ \neg\neg(x = 0)$ while the generic local ring does not.

Hence it is useful to turn to geometric theories which refer to a given mathematical object. For instance, given a ring A, there is the theory of local localizations of A, and its classifying topos is known in algebraic geometry as the little Zariski topos of A, the topos of sheaves over the spectrum of A. If A is reduced, this topos validates the dual field condition that any noninvertible element is zero. This property has been used to give a short and even constructive proof of Grothendieck's generic freeness lemma, substantially improving on previously published proofs [8].

In time, further nongeometric sequents holding in the big Zariski topos of an arbitrary base scheme have been found [9, Section 18.4]. These include:

- $\underline{\mathbb{A}}^1$ is *anonymously algebraically closed* in the sense that any monic polynomial $p : \underline{\mathbb{A}}^1[T]$ of degree at least one does *not not* have a zero.
- The Nullstellensatz holds: Let $f_1, \ldots, f_m \in \underline{\mathbb{A}}^1[X_1, \ldots, X_n]$ be polynomials without a common zero in $(\underline{\mathbb{A}}^1)^n$. Then there are polynomials $g_1, \ldots, g_m \in \underline{\mathbb{A}}^1[X_1, \ldots, X_n]$ such that $\sum_i g_i f_i = 1$.
- Any function $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is given by a unique polynomial.
- $\underline{\mathbb{A}}^1$ is microaffine: Let $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1 \mid \varepsilon^2 = 0\}$. Let $f : \Delta \to \underline{\mathbb{A}}^1$ be an arbitrary function. Then there are unique elements $a, b : \underline{\mathbb{A}}^1$ such that $f(\varepsilon) = a + b\varepsilon$ for all $\varepsilon : \Delta$.

- $\underline{\mathbb{A}}^1$ is *synthetically quasicoherent*: For any finitely presentable $\underline{\mathbb{A}}^1$-algebra A, the canonical homomorphism $A \to (\underline{\mathbb{A}}^1)^{\mathrm{Spec}(A)}$, where $\mathrm{Spec}(A)$ is defined as the set of $\underline{\mathbb{A}}^1$-algebra homomorphisms $A \to \underline{\mathbb{A}}^1$, is bijective.

All of these nongeometric sequents are useful for the purposes of synthetic algebraic geometry, the desire to carry out algebraic geometry in a language close to the simple language of the 19th and the beginning of the 20th century while still being fully rigorous and fully general, working over arbitrary base schemes instead of restricting to the field of complex numbers.

Nontrivial nongeometric sequents are not an exclusive feature of sheaf toposes. The generic model of a theory of presheaf type typically also validates such sequents. For instance, all of the aforementioned properties of the generic local ring are shared by the generic ring, which lives in the presheaf topos $[\mathrm{Ring}_{\mathrm{fp}}, \mathrm{Set}]$.

Characterizing nongeometric sequents. Referring to the field condition in the little Zariski topos, Tierney remarked around the time that those sequents were first studied that "[it] is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas" [21, p. 209]. In view of their importance, is there a way to discover nongeometric sequents in a systematic fashion? To characterize the nongeometric sequents holding in classifying toposes? To this end, Wraith put forward a specific conjecture [24, p. 336]:

> *The problem of characterising all the non-geometric properties of a generic model appears to be difficult. If the generic model of a geometric theory $\mathbb{T}$ satisfies a sentence α then any geometric consequence of $\mathbb{T} + \alpha$ has to be a consequence of $\mathbb{T}$. We might call α $\mathbb{T}$-redundant. Does the generic $\mathbb{T}$-model satisfy all $\mathbb{T}$-redundant sentences?*

Because classical logic is conservative over intuitionistic logic for geometric sequents, this question has a trivial negative answer: No, any instance of the law of excluded middle over the signature of $\mathbb{T}$ is $\mathbb{T}$-redundant but typically not validated by the generic $\mathbb{T}$-model. Moreover, Bezem, Buchholtz and Coquand recently showed that the answer is still negative even if appropriate care is taken to exclude these counterexamples [3]. Hence our characterization of the first-order theory validated by the generic $\mathbb{T}$-model is necessarily more nuanced.

Our starting point was the empirical observation [9, p. 164] that in the case of the big Zariski topos, every true known nongeometric sequent followed from just a single such, namely the synthetic quasicoherence of the generic model, and in earlier work we surmised that one could formulate an appropriate metatheorem explaining this observation and generalizing it to arbitrary classifying toposes [9, Speculation 22.1]. This hope turned out to be true, in the sense we will now indicate.

A general Nullstellensatz. To explain the relevant background, the somewhat vague question "to which extent does the classifying topos $\mathrm{Set}[\mathbb{T}]$ realize that it is the classifying topos for $\mathbb{T}$?" is useful as a guiding principle. This is easiest to visualize with a concrete example for $\mathbb{T}$, such as the theory of rings.

Let A be a ring. A simple version of the classical Nullstellensatz states: *For any polynomials f and g over A, if any zero of f is also a zero of g, then there is a polynomial h such that $g = hf$.* The polynomial h can be regarded as an "algebraic certificate" of the hypothesis. This principle holds for instance in the case that A is an algebraically closed field and g is the unit polynomial. We will see below that it is also true, without any restriction on g, for the generic ring.

We could try to generalize the Nullstellensatz to arbitrary geometric theories $\mathbb{T}$ as follows: *For any geometric sequent σ, if σ holds for a given $\mathbb{T}$-model M then σ is provable modulo $\mathbb{T}$.* In place of the algebraic certificate we now have a logical certificate, a *proof* of σ.

However, this generalized statement is typically false, even for the generic model $U_{\mathbb{T}}$: The statement

$$\begin{aligned}\mathrm{Set}[\mathbb{T}] \models \ulcorner &\text{for any geometric sequent } \sigma, \\ &\qquad \text{if } \sigma \text{ holds for } U_{\mathbb{T}} \text{ then } \underline{\mathbb{T}} \text{ proves } \sigma\urcorner\end{aligned}$$

does not hold.[2] In this sense $\mathrm{Set}[\mathbb{T}]$ does not believe that $U_{\mathbb{T}}$ is the generic $\underline{\mathbb{T}}$-model.

A concrete counterexample is as follows. Let $\mathbb{T}$ be the theory of rings and let σ be the sequent $(\top \vdash 1 + 1 = 0)$. Since there is an intuitionistic proof that $\mathbb{T}$ does not prove σ and toposes are sound with respect to intuitionistic logic, the statement $\ulcorner\underline{\mathbb{T}}$ proves $\sigma\urcorner$ is false from the internal point of view of $\mathrm{Set}[\mathbb{T}]$. However, it is not the case that the statement $\ulcorner 1+1 = 0$ in $U_{\mathbb{T}}\urcorner$ is false from the internal point of view. In fact, this statement holds in a nontrivial slice of $\mathrm{Set}[\mathbb{T}]$, the open subtopos coinciding with the classifying topos of the theory of rings of characteristic two.

Intuitively, the problem is that while the meaning of $\ulcorner\underline{\mathbb{T}}$ proves $\sigma\urcorner$ is fixed, the meaning of $\ulcorner\sigma$ holds for $U_{\mathbb{T}}\urcorner$ varies with the slice, as $U_{\mathbb{T}}$ shifts shape on different stages. This mismatch is solved by passing from $\underline{\mathbb{T}}$ to a varying theory, the internal theory $\underline{\mathbb{T}}/U_{\mathbb{T}}$ defined in Section 3. If $\mathbb{T}$ is the theory of rings, then $\underline{\mathbb{T}}/U_{\mathbb{T}}$ is the $\mathrm{Set}[\mathbb{T}]$-theory of $U_{\mathbb{T}}$-algebras. Unlike $\underline{\mathbb{T}}$, this theory is not the pullback of an external geometric theory. We then have, subject to some qualifications made precise in Section 3, the following general Nullstellensatz:

Theorem 1.1. *Let $\mathbb{T}$ be a geometric theory. Then, internally to* $\mathrm{Set}[\mathbb{T}]$*:*

$$\text{\textit{A geometric}}^* \text{ \textit{sequent} } \sigma \text{ \textit{holds for} } U_{\mathbb{T}} \text{ \textit{if and only if} } \underline{\mathbb{T}}/U_{\mathbb{T}} \text{ \textit{proves} } \sigma. \qquad (\dagger)$$

To illustrate Theorem 1.1, let $\mathbb{T}$ be the theory of rings and let σ be the sequent $(f(x) = 0 \vdash_x g(x) = 0)$ for some polynomials f and g. To say that σ holds for $U_{\mathbb{T}}$ amounts to saying that any zero $x : U_{\mathbb{T}}$ of f is also a zero of g, and to say that $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves σ amounts to saying that in $U_{\mathbb{T}}[X]/(f(X))$, the free $U_{\mathbb{T}}$-algebra on one generator X subject to the relation $f(X) = 0$, the relation $g([X]) = 0$ holds. Hence we obtain an algebraic Nullstellensatz:

$$\mathrm{Set}[\mathbb{T}] \models \forall f, g : U_{\mathbb{T}}[X].\ \big((\forall x : U_{\mathbb{T}}.\ f(x) = 0 \Rightarrow g(x) = 0) \Longleftrightarrow \exists h : U_{\mathbb{T}}[X].\ g = hf\big).$$

The statement ($\dagger$) is not a geometric sequent. Therefore it is not to be expected that it passes from $\mathrm{Set}[\mathbb{T}]$ to a subtopos $\mathrm{Set}[\mathbb{T}']$ corresponding to a quotient theory $\mathbb{T}'$ of $\mathbb{T}$, and indeed in general it does not. However, there is still a useful substitute,

[2]Here $\underline{\mathbb{T}}$ is the internal geometric theory induced by $\mathbb{T}$, obtained by pulling back the set of sorts, the set of function symbols and so on along the geometric morphism $\mathrm{Set}[\mathbb{T}] \to \mathrm{Set}$. For instance, if $\mathbb{T}$ is the theory of rings, then from the internal point of view of $\mathrm{Set}[\mathbb{T}]$ the theory $\underline{\mathbb{T}}$ will again be the theory of rings. More details will be given in Section 2.3. The corner quotes indicate that for sake of readability, the translation into formal language is to be carried out by the reader.

The displayed statement is much stronger than the statement that for any geometric sequent σ, if $\mathrm{Set}[\mathbb{T}] \models \ulcorner\sigma$ holds for $U_{\mathbb{T}}\urcorner$ then $\mathbb{T}$ proves σ. This latter statement, where the universal quantifier and the "if ... then" have been pulled out, is true.

which we formulate as Corollary 3.9. This substitute broadens the scope of the Nullstellensatz and is, whenever applicable, useful for simplifying computations.

Summarizing, the situation is as follows.

- The generic model $U_{\mathbb{T}}$ is a conservative $\mathbb{T}$-model.
- The topos Set[$\mathbb{T}$] does not believe that $U_{\mathbb{T}}$ is a conservative $\underline{\mathbb{T}}$-model.
- The topos Set[$\mathbb{T}$] does believe that $U_{\mathbb{T}}$ is a conservative* $\underline{\mathbb{T}}/U_{\mathbb{T}}$-model.

Theorem 1.1 is a source of nongeometric sequents. Indeed, it is the universal such source in the sense that any first-order formula which holds for $U_{\mathbb{T}}$ can be deduced from (†):

Theorem 1.2. *Let $\mathbb{T}$ be a geometric theory. Let φ be a first-order formula over the signature of $\mathbb{T}$. Then the following statements are equivalent.*

(1) *The formula φ holds for $U_{\mathbb{T}}$.*
(2) *The formula φ is provable in first-order intuitionistic logic modulo the axioms of $\mathbb{T}$ and the additional axiom (†).*

It is not entirely obvious how to correctly formalize axiom (†) in unadorned first-order logic, and in fact, we do not believe that an entirely faithful formalization is possible. We will resolve this issue in the body of this note, where we will restate Theorem 1.2 in a more precise form (Theorem 3.12).

Theorem 1.2 characterizes the first-order formulas which hold for the generic model. We could of course wish for a more explicit characterization; but since even the characterization of geometric sequents holding for the generic model (they are precisely those which are provable in geometric logic modulo $\mathbb{T}$) is of a rather implicit nature, this wish appears unfounded.

We stress that our characterization is more explicit than the tautologous characterization ("a first-order formula holds for $U_{\mathbb{T}}$ iff it is provable modulo $\mathbb{T}'$, where $\mathbb{T}'$ is the first-order theory whose set of axioms is the set of first-order formulas satisfied by $U_{\mathbb{T}}$") and the (incorrect) characterization "a first-order formula holds for $U_{\mathbb{T}}$ iff it is $\mathbb{T}$-redundant". Indeed, if $\mathbb{T}$ happens to be coherent and recursively axiomatizable, then in stating Theorem 1.2 we may restrict to coherent existential fixed-point logic, and the resulting theory will again be recursively axiomatizable.

Related work. The topos-theoretic Nullstellensatz is related to several precursors. A corollary of the Nullstellensatz is that, over the first-order theory validated by $U_{\mathbb{T}}$, any first-order formula is in fact logically equivalent to a geometric formula. This corollary has already been observed by Butz and Johnstone [11, Lemma 4.2]. At that point, a characterization of the first-order formulas in the general case, of the form as in Theorem 1.2, was still missing.

The higher-order variant of our Nullstellensatz (Theorem 5.2) relativizes Caramello's completeness theorem [13, Theorem 2.4(ii)]. Her theorem is the external statement that any subobject of $U_{\mathbb{T}}$ is given by the interpretation of a geometric formula over the signature of $\mathbb{T}$; our relativization states that, internally to Set[$\mathbb{T}$], any subset of $U_{\mathbb{T}}$ is given by a geometric* formula over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$. As in the first-order case, the passage from the external to the internal phrasing necessitates the switch from $\mathbb{T}$ to $\underline{\mathbb{T}}/U_{\mathbb{T}}$.

The Nullstellensatz yields for a given (perhaps conditional) truth about the generic model a proof modulo the axioms of $\underline{\mathbb{T}}/U_{\mathbb{T}}$. As illustrated in Section 6, the procedure we typically use to extract fruitful information from such a proof is to apply it to a further $\underline{\mathbb{T}}/U_{\mathbb{T}}$-model which we specifically construct for the purposes at hand. Hence we go from truth in $U_{\mathbb{T}}$ via provability to truth in a further model, that is, we use provability as a (one-way) *bridge*:

$$\begin{tikzcd}[column sep=small] & \underline{\mathbb{T}}/U_{\mathbb{T}} \text{ proves } \sigma \arrow[dr, Rightarrow, bend left] & \\ \sigma \text{ holds for } U_{\mathbb{T}} \arrow[ur, Leftrightarrow, bend left] & & \sigma \text{ holds for } M \end{tikzcd}$$

This perspective is inspired by Caramello's research program [12], though it is not an instance of her main technical device for establishing bridges, namely exploiting the fact that a single topos may admit descriptions using quite different sites.

Outlook. The Nullstellensatz yields a description of the first-order and higher-order theory validated by the generic model of a geometric theory. As already indicated, with more examples and details to be given in Section 6, this description explains and puts into perspective several established results and observations.

From the point of view of applications, the main task for the future is to explore the description in the case of particular geometric theories of interest. Just as the reduction technique "any reduced ring is a field" – valid because the generic localization of a reduced ring is a field – enabled a short and simple proof of Grothendieck's generic freeness lemma [9, Section 11.5], more reduction techniques along these lines should exist in a wide range of subjects. We hope that with the Nullstellensatz at hand, the discovery of such techniques can progress in a more systematic fashion.

From the point of view of theory, the appropriate context for the Nullstellensatz should be determined. We formulate the Nullstellensatz in the context of geometric theories and their classifying Grothendieck toposes, but there should be analogues of the Nullstellensatz in other contexts where it makes sense to speak of "classifying gadgets", and these analogues should shed light on the topos-theoretic version and ideally even explain it from deeper principles. In particular, preliminary computations indicate that there is a version of the Nullstellensatz for arithmetic universes, the predicative cousin of toposes introduced by Joyal and recently an important object of consideration by Maietti and Vickers [17, 18, 23].

Outline. In Section 2, we review background on the internal language of toposes, classifying toposes and internal geometric theories. Section 3 contains proofs of the main theorems in the full generality of geometric theories. Restricting to Horn theories allows for a treatment which is more algebraic and less logical in flavor. For the benefit of readers with a more algebraic background, we include a mostly self-contained account of the Horn case as Section 4. We generalize our main theorems to the higher-order case in Section 5 and conclude with applications in Section 6.

Throughout we work in a constructive metatheory, to allow our results to be interpreted internally to toposes.

Acknowledgments. XXX

2. Background

2.1. **Background on the internal language of Grothendieck toposes.**

2.2. **Background on classifying toposes.** We use the usual convention of abbreviating "$x_1 : X_1, \ldots, x_n : X_n$" as "$\vec{x} : \vec{X}$" or even just "$\vec{x}$".

Definition 2.1. The *syntactic site* $\mathcal{C}_\mathbb{T}$ of a geometric theory $\mathbb{T}$ has:

(1) as objects *geometric formulas in contexts* $\{x_1 : X_1, \ldots, x_n : X_n.\ \varphi\}$ where φ is a geometric formula over the signature of $\mathbb{T}$ in the displayed context;
(2) as set $\mathrm{Hom}_{\mathcal{C}_\mathbb{T}}(\{\vec{x}.\ \varphi\}, \{\vec{y}.\ \psi\})$ of morphisms the set of formulas θ in the context $\vec{x}, \vec{y}$ which are $\mathbb{T}$-provably functional, modulo $\mathbb{T}$-provable equivalence of such formulas;
(3) as covering families those families $(\{\vec{x}_i.\ \varphi_i\} \xrightarrow{\theta_i} \{\vec{y}.\ \psi\})_i$ for which $\mathbb{T}$ proves the sequent $(\psi \vdash_{\vec{y}} \bigvee_i \exists \vec{x}_i.\ \theta_i)$.

Definition 2.2. The *classifying topos* $\mathrm{Set}[\mathbb{T}]$ of a geometric theory $\mathbb{T}$ is the topos of set-valued sheaves on $\mathcal{C}_\mathbb{T}$.

Writing $よ : \mathcal{C}_\mathbb{T} \to \mathrm{Set}[\mathbb{T}]$ for the Yoneda embedding,[3] the *generic model* $U_\mathbb{T}$ of $\mathbb{T}$ interprets a sort X of $\mathbb{T}$ as the sheaf $よ\{x : X.\ \top\}$, a function symbol $f : X_1 \cdots X_n \to Y$ as the morphism given by the $\mathbb{T}$-provably functional formula $f(x_1, \ldots, x_n) = y$ and a relation symbol $R \rightarrowtail X_1 \cdots X_n$ by the subobject $よ\{\vec{x}.\ R(\vec{x})\} \rightarrowtail よ\{\vec{x}.\ \top\}$.

Remark 2.3. Sections of the sheaf $よ\{y : Y.\ \top\}$ over a stage $A = \{\vec{x}.\ \varphi\} \in C_\mathbb{T}$ are in one-to-one correspondence with "unique descriptions" over A, that is formulas θ in the context $\vec{x}, y$ such that $\mathbb{T}$ proves $(\varphi \vdash_{\vec{x}} \exists! y : Y.\ \theta)$, up to provable equivalence.

Theorem 2.4. *The generic model is universal in the sense that for any Grothendieck topos $\mathcal{E}$, the functor*

$$\textit{(category of geometric morphisms } \mathcal{E} \to \mathrm{Set}[\mathbb{T}]\textit{)} \longrightarrow \textit{(category of } \mathbb{T}\textit{-models in } \mathcal{E}\textit{)}$$

given by $f \mapsto f^* U_\mathbb{T}$ is an equivalence of categories.

Proof. See, for instance, [12, Theorem 2.1.8] or [15, discussion before Proposition D3.1.12]. ☐

Proposition 2.5. *Let α and φ be geometric formulas in a context $\vec{x}$ over the signature of a geometric theory $\mathbb{T}$. Then the following statements are equivalent:*

(1) $\mathrm{Set}[\mathbb{T}] \models \forall \vec{x}.\ (\alpha \Rightarrow \varphi)$.
(2) $\{\vec{x}.\ \alpha\} \models \varphi$, *where the free variables in φ are interpreted as their generic values over $\{\vec{x}.\ \alpha\}$, that is as the projection maps $\{\vec{x}.\ \alpha\} \to \{x_i : X_i.\ \top\}$.*
(3) $\mathbb{T}$ *proves* $(\alpha \vdash_{\vec{x}} \varphi)$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows immediately by unrolling the Kripke–Joyal semantics. The equivalence (2) $\Leftrightarrow$ (3) is by induction on the structure of φ. ☐

2.3. **Background on internal geometric theories.** The notions of signatures, geometric theories and classifying toposes can be relativized to the internal world of arbitrary toposes with natural numbers objects. Basics on internal signatures, internal geometric theories and internal classifying toposes are folklore [24, p. 334]; a careful treatment is due to Shawn Henry [14].

Briefly, an internal signature Σ internal to a topos $\mathcal{E}$ consists of an object of sorts, an object of function symbols, an object of relation symbols, and various morphisms indicating the sorts involved with the function and relation symbols.

[3]With this notational choice we are following Riehl and Verity [20].

Given an internal signature Σ internal to a Grothendieck topos $\mathcal{E}$ (or elementary topos with a natural numbers object), we can successively build the object of contexts (the object of lists of sorts); the object of terms (equipped with a morphism to the object of contexts); the object of atomic propositions (again equipped with such a morphism); the object of geometric formulas (again so); and the object of geometric sequents (again so). An internal geometric theory $\mathbb{T}$ over Σ is then given by a subobject of the object of geometric sequents, interpreted as the object of axioms of $\mathbb{T}$. Given such an internal theory $\mathbb{T}$, we can build the subobject of the provable sequents.

For the most part, these objects can be obtained by simply carrying out the familiar constructions of the set of contexts, of the set of terms and so on in the internal language of $\mathcal{E}$. Some care is required in constructing the object of geometric formulas: Inductively closing the object of basic propositions under the connectives of geometric logic will fail because of size issues – even if we carry this out in Set, we will end up with a proper class. Instead the object of geometric formulas should be constructed so as to only contain geometric formulas in canonical form ("$\bigvee \exists \cdots \exists.\ \varphi_1 \wedge \cdots \wedge \varphi_n$" for atomic propositions φ_i). There is no real loss in this restriction since in foundations where it makes sense to say this, any geometric formula is provably equivalent to a geometric formula in canonical form.

Similarly, we cannot construct the subobject of provable sequents by first constructing an object of raw (arbitrarily-branching) trees and then cutting down to an object of correctly formed proof trees. Instead, the subobject of provable sequents can be obtained by intersecting, internally to $\mathcal{E}$, all subsets of the set of sequents which are closed under the rules of geometric logic.[4]

None of these complications arise when setting up the theory of internal coherent theories, where disjunctions are restricted to be finitary.

Example 2.6. An ordinary geometric theory is the same as a geometric theory internal to the topos Set. A geometric sequent is provable in the ordinary sense if and only if, from the internal point of view of Set, it is contained in the object of provable sequents.

Example 2.7. An ordinary geometric theory $\mathbb{T}$ over an ordinary signature Σ can be pulled back along a geometric morphism $\mathcal{E} \to \mathrm{Set}$ to yield an internal geometric theory $\underline{\mathbb{T}}$ over the internal signature $\underline{\Sigma}$ in $\mathcal{E}$. The object of sorts of $\underline{\Sigma}$ is the pullback of the set of sorts of Σ, the object of function symbols of $\underline{\Sigma}$ is the pullback of the set of function symbols of Σ, and so on.

Disjunctions appearing in internal geometric formulas may be indexed by arbitrary objects of the topos, just like disjunctions appearing in ordinary external geometric formulas over an ordinary signature may be indexed by arbitrary sets. If Σ is an internal signature in a Grothendieck topos $\mathcal{E}$, the object of geometric formulas over Σ has an important subobject, the subobject of those formulas such that locally, any appearing disjunction is indexed by a constant sheaf. Such internal geometric formulas will be called *geometric** *formulas.*

[4]Equivalently one can appeal to the Knaster–Tarski fixed point theorem, which is constructively valid in the form we require here [2], to construct the least fixed point of the operator which, given a set M of geometric sequents, computes the set of geometric sequents which can be derived from those in M in at most one step. Details on how to carry out such kinds of inductive constructions can be found in [7] and more specifically in [14, Chapter III].

Remark 2.8. A priori, there are two different notions of what it could mean that a geometric* formula is provable: It could be provable when regarded as a geometric formula, or we could allow only geometric* formulas as intermediate formulas in proofs. One can show that these two notions coincide, similarly to how a coherent sequent is provable in geometric logic if and only if it is provable in coherent logic. Since we do not require this fact in the course of this note, we omit a detailed verification.

3. The first-order Nullstellensatz

Given a geometric theory $\mathbb{T}$ with its generic model $U_\mathbb{T}$ in $\mathrm{Set}[\mathbb{T}]$, our main theorems will reference a certain internal theory $\underline{\mathbb{T}}/U_\mathbb{T}$ internal to $\mathrm{Set}[\mathbb{T}]$. This theory is defined as follows.

Definition 3.1. Let $\mathbb{T}$ be a geometric theory. The theory $\underline{\mathbb{T}}/U_\mathbb{T}$ is the geometric theory internal to $\mathrm{Set}[\mathbb{T}]$ which arises from the pulled-back theory $\underline{\mathbb{T}}$ by adding additional constant symbols e_x of the appropriate sorts, one for each element $x : U_\mathbb{T}$, axioms $(\top \vdash f(e_{x_1}, \ldots, e_{x_n}) = e_{f(x_1,\ldots,x_n)})$ for each function symbol f and n-tuple of elements of $U_\mathbb{T}$ (of the appropriate sorts), and axioms $(\top \vdash R(e_{x_1}, \ldots, e_{x_n}))$ for each relation symbol R and n-tuple $(x_1, \ldots, x_n)$ (of the appropriate sorts) such that $R(x_1, \ldots, x_n)$.

From the point of view of $\mathrm{Set}[\mathbb{T}]$, a model of $\underline{\mathbb{T}}/U_\mathbb{T}$ is a model of $\underline{\mathbb{T}}$ equipped with a $\underline{\mathbb{T}}$-homomorphism from $U_\mathbb{T}$. In particular, the identity $(U_\mathbb{T} \to U_\mathbb{T})$ is a model of $\underline{\mathbb{T}}/U_\mathbb{T}$. This is what we mean when we say that $U_\mathbb{T}$ is in a canonical way a $\underline{\mathbb{T}}/U_\mathbb{T}$-model.

Example 3.2. Let $\mathbb{T}$ be the theory of rings. Then $\underline{\mathbb{T}}/U_\mathbb{T}$ is, from the internal point of view of $\mathrm{Set}[\mathbb{T}]$, the theory of $U_\mathbb{T}$-algebras.

Example 3.3. Let $\mathbb{T}$ be a geometric theory. Let M be a model of $\mathbb{T}$ in the category of sets. Let $f : \mathrm{Set} \to \mathrm{Set}[\mathbb{T}]$ be the corresponding geometric morphism. Then $f^*(\underline{\mathbb{T}}/U_\mathbb{T})$ is the theory of M-algebras ($\mathbb{T}$-models equipped with a $\mathbb{T}$-homomorphism from M). This is because $f^*\underline{\mathbb{T}} = \mathbb{T}$, $f^*U_\mathbb{T} = M$ and because the construction of the theory $\underline{\mathbb{T}}/U_\mathbb{T}$ is geometric.

Remark 3.4. From the internal point of view of $\mathrm{Set}[\mathbb{T}]$, we can construct the classifying topos of $\underline{\mathbb{T}}/U_\mathbb{T}$. Externally, this construction gives rise to a bounded topos over $\mathrm{Set}[\mathbb{T}]$, hence to a Grothendieck topos. Using for instance the technique described in [10], one can show that this topos classifies the theory of homomorphisms between $\mathbb{T}$-models. This topos can also be obtained as the lax pullback $(\mathrm{Set}[\mathbb{T}] \Rightarrow_{\mathrm{Set}[\mathbb{T}]} \mathrm{Set}[\mathbb{T}])$. There are two canonical geometric morphisms from this topos to $\mathrm{Set}[\mathbb{T}]$, the morphism computing the domain and the morphism computing the codomain; the morphism obtained by externalizing the internal construction is the former.

Lemma 3.5. *Let $\mathbb{T}$ be a geometric theory. Let α be a geometric formula over the signature of $\mathbb{T}$ in a context $x_1 : X_1, \ldots, x_n : X_n$. Then*

$$\{\vec{x}.\ \alpha\} \models \ulcorner \underline{\mathbb{T}}/U_\mathbb{T} \text{ proves } (\top \vdash_{[]} \alpha)\urcorner,$$

where the free variables $\vec{x}$ occurring in α are first interpreted as in Proposition 2.5 and then regarded as the induced constant symbols provided by the enlarged signature of $\underline{\mathbb{T}}/U_\mathbb{T}$.

Proof. By induction on the structure of α. The cases of "$\top$" and "$\wedge$" are trivial; the cases of "$\bigvee$" and "$\exists$" follow from passing to suitable coverings; and the case of atomic propositions is by definition of $\underline{\mathbb{T}}/U_{\mathbb{T}}$. □

Lemma 3.6. *Let $\mathbb{T}$ be a geometric theory. Let φ be a section of the sheaf of geometric* formulas over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$ over a stage $A \in \mathcal{C}_{\mathbb{T}}$. Then there is a covering $(A_i \to A)_i$ of A such that for each index i, there is a formula φ_i over the signature of $\mathbb{T}/U_{\mathbb{T}}(A_i)$ such that $A_i \models \ulcorner\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves $(\varphi \dashv\vdash \varphi_i)\urcorner$.*

Proof. By passing to a covering, we may suppose that φ is given by an (external) geometric formula over the signature of $\underline{\mathbb{T}}(A)/U_{\mathbb{T}}(A)$.

Any function symbol and relation symbol of $\underline{\mathbb{T}}(A)$ occurring in φ is locally given by a symbol of $\mathbb{T}$. Hence the claim would be trivial if φ were a coherent formula, for in this case we would just have to pass to further coverings, one for each occurring symbol, a finite number of times in total.

However, in general, we cannot conclude as easily. Write $A = \{\vec{x}.\ \alpha\}$. Let R be a relation symbol of $\underline{\mathbb{T}}(A)$ occurring in φ. By the explicit description of constant sheaves as sheaves of locally constant maps, there is a covering $(\{\vec{y}_j.\ \alpha_j\} \xrightarrow{[\theta_j]} \{\vec{x}.\ \alpha\})_j$ such that, restricted to $\{\vec{y}_j.\ \alpha_j\}$, R is given by a relation symbol R_j of $\mathbb{T}$. To construct the desired formula φ', we replace any such occurrence $R(\ldots)$ in φ by

$$\bigvee_j \big((\exists \vec{y}_j.\ \theta_j) \wedge R_j(\ldots)\big).$$

(The formulas θ_j will typically contain instances of the variables $\vec{x}$. When writing down this replacement, we treat these as in Lemma 3.5. Hence this replacement is set in the same context as φ, only that new constant symbols might occur.) In a similar vein we treat any occurrence of function symbols.

The resulting formula φ' is a geometric formula over the signature of $\mathbb{T}/U_{\mathbb{T}}(A)$. The verification of $A \models \ulcorner\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves $(\varphi \dashv\vdash \varphi')\urcorner$ rests on the observation

$$A \models \ulcorner\underline{\mathbb{T}}/U_{\mathbb{T}} \text{ proves } \big((\exists \vec{y}_k.\ \theta_k) \vdash_{[]} \bigvee\{\top \mid (\exists \vec{y}_k.\ \theta_k) \text{ holds for } U_{\mathbb{T}}\}\big)\urcorner$$

which in turn can be checked on the covering $(\{\vec{y}_j.\ \alpha_j\} \xrightarrow{[\theta_j]} \{\vec{x}.\ \alpha\})_j$, applying Lemma 3.5 and using that $\mathbb{T}$ (and hence $\underline{\mathbb{T}}$) proves $((\exists \vec{y}_j.\ \theta_j) \wedge (\exists \vec{y}_k.\ \theta_k) \vdash_{\vec{x}} \bigvee\{\top \mid j = k\})$. (This kind of reasoning also appears, for instance, in [5, p. 20].) □

Theorem 3.7. *Let $\mathbb{T}$ be a geometric theory. Then, internally to* Set[$\mathbb{T}$]*, for any geometric* sequent σ over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$, the following statements are equivalent:*

(1) *The sequent σ holds for $U_{\mathbb{T}}$.*
(2) *The sequent σ is provable modulo $\underline{\mathbb{T}}/U_{\mathbb{T}}$.*

Proof. The direction (2) $\Rightarrow$ (1) is immediate because $U_{\mathbb{T}}$ is, from the internal point of view of Set[$\mathbb{T}$], a $\underline{\mathbb{T}}/U_{\mathbb{T}}$-model.

For the direction (1) $\Rightarrow$ (2) we have to verify that, given any stage $A \in \mathcal{C}_{\mathbb{T}}$ and any section σ of the sheaf of geometric* sequents over A, if $A \models \ulcorner\sigma$ holds for $U_{\mathbb{T}}\urcorner$ then $A \models \ulcorner\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves $\sigma\urcorner$. By Lemma 3.6 we may suppose that σ is an (external) geometric sequent over the signature of $\mathbb{T}/U_{\mathbb{T}}(A)$.

Writing $A = \{\vec{x}.\ \alpha\}$ and $\sigma = (\varphi \vdash_{\vec{y}} \psi)$, we have $\{\vec{x}.\ \alpha\} \models \forall \vec{y}.\ (\varphi \Rightarrow \psi)$ by assumption, hence $\{\vec{x}, \vec{y}.\ \alpha \wedge \varphi\} \models \psi$ (where we inline any occurence of an element of $U_{\mathbb{T}}(A)$ as a constant symbol in φ, recalling Remark 2.3). Thus $\mathbb{T}$ proves $(\alpha \wedge \varphi \vdash_{\vec{x},\vec{y}} \psi)$ (where we now have to do the same inlining for ψ as well). This proof can be pulled back from Set to $\mathrm{Set}[\mathbb{T}]/\!\!\not\!\!\!\!\!\!\text{\textsterling}A$ to obtain $A \models \ulcorner \underline{\mathbb{T}}/U_{\mathbb{T}}$ proves $(\alpha \wedge \varphi \vdash_{\vec{x},\vec{y}} \psi)\urcorner$. By Lemma 3.5, we also have $A \models \ulcorner \underline{\mathbb{T}}/U_{\mathbb{T}}$ proves $(\top \vdash_{[]} \alpha)\urcorner$ (where the free variables occurring in α are interpreted as the generic values available over A), hence $A \models \ulcorner \underline{\mathbb{T}}/U_{\mathbb{T}}$ proves $(\varphi \vdash_{\vec{y}} \psi)\urcorner$. □

The force of the Nullstellensatz of Theorem 3.7 is for sequents σ which are not of the form $(\top \vdash_{[]} \varphi)$. Indeed, for those special sequents, the implication (σ holds for $U_{\mathbb{T}} \Rightarrow \underline{\mathbb{T}}/U_{\mathbb{T}}$ proves $(\top \vdash_{[]} \varphi)$) can be proven by a simple induction on the structure of φ. Only for more general sequents does Theorem 3.7 express a nontrivial fact: that truth of an universally-quantified conditional statement implies a single unquantified unconditional statement.

Remark 3.8. The restriction in Theorem 3.7 to geometric* sequents cannot be lifted, that is the generalization of Theorem 3.7 to arbitrary internal geometric sequents is false. For instance, in the case that $\mathbb{T}$ is the theory of objects, the internal geometric sequent $(\top \vdash_{x\,:\,U_{\mathbb{T}}} \bigvee_{a\,:\,U_{\mathbb{T}}}(x = e_a))$ trivially holds for $U_{\mathbb{T}}$. However, this sequent is not provable modulo $\underline{\mathbb{T}}/U_{\mathbb{T}}$, as for instance the model $U_{\mathbb{T}} \amalg \{\star\}$ does not validate it.

Corollary 3.9. *Let $\mathbb{T}$ be a geometric theory. Let $\mathbb{T}'$ be a quotient theory of $\mathbb{T}$. Assume that the generic model $U_{\mathbb{T}}$ is a sheaf for the topology on* $\mathrm{Set}[\mathbb{T}]$ *cutting out the subtopos* $\mathrm{Set}[\mathbb{T}']$*. Then the following statement holds internally to* $\mathrm{Set}[\mathbb{T}']$*:*

A geometric* sequent σ with Horn consequent holds for $U_{\mathbb{T}'}$ iff $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves σ.

Proof. In general, the generic model of $\mathbb{T}'$ is the pullback of the generic model of $\mathbb{T}$ to the subtopos $\mathrm{Set}[\mathbb{T}']$ [13, Lemma 2.3]. By the sheaf assumption, the objects $U_{\mathbb{T}'}$ and $U_{\mathbb{T}}$ actually agree, that is $U_{\mathbb{T}}$ is contained in the subtopos and has the universal property of $U_{\mathbb{T}'}$.

The "if" direction is trivial, as $U_{\mathbb{T}'}$ is a $\underline{\mathbb{T}}/U_{\mathbb{T}}$-model.

For the "only if" direction, we use that a statement holds in $\mathrm{Set}[\mathbb{T}']$ if and only if its ∇-translation holds in $\mathrm{Set}[\mathbb{T}]$, where ∇ is the modal operator associated to the topology cutting out $\mathrm{Set}[\mathbb{T}']$ [9, Theorem 6.31]. Exploiting some of the simplification rules of the ∇-translation [9, Section 6.6], it hence suffices to verify, internally to $\mathrm{Set}[\mathbb{T}]$, that:

For any geometric* sequent $\sigma = (\varphi \vdash_{\vec{x}} \psi)$ where ψ is a Horn formula,
if $\forall x_1, \ldots, x_n : U_{\mathbb{T}}.\ (\varphi \Rightarrow \nabla\psi)$, then $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves σ.

Since ∇ commutes with finite conjunctions and since the sheaf assumption implies that $\nabla(s = t)$ is equivalent to $s = t$ and that, for relation symbols R, the statement $\nabla(R(s_1, \ldots, s_m))$ is equivalent to $R(s_1, \ldots, s_m)$, the statement $\nabla\psi$ is equivalent to ψ. Hence the claim follows from Theorem 3.7. □

A situation in which the sheaf assumption of Corollary 3.9 is satisfied is when $\mathbb{T}$ is a Horn theory and the topology cutting out $\mathrm{Set}[\mathbb{T}']$ is subcanonical. For instance, this is the case if $\mathbb{T}$ is the theory of rings and $\mathrm{Set}[\mathbb{T}']$ is one of several well-known toposes in algebraic geometry such as the big Zariski topos, the big étale topos or the big fppf topos.

Remark 3.10. In the case that the subtopos $\mathrm{Set}[\mathbb{T}']$ is a dense subtopos of $\mathrm{Set}[\mathbb{T}]$, Corollary 3.9 can be strengthened to allow $\perp$ as consequent, since in this case we have $(\nabla\psi) \Rightarrow \psi$ also for $\psi \equiv \perp$.

Theorem 3.7 cannot be strengthened to arbitrary first-order (or first-order* or even finitary first-order) formulas in place of geometric* sequents. For instance, in the case that $\mathbb{T}$ is the theory of rings, the generic model $U_\mathbb{T}$ validates the finitary first-order formula $\ulcorner$any element x for which $(x = 0 \Rightarrow 1 = 0)$ is invertible$\urcorner$, but $\underline{\mathbb{T}}/U_\mathbb{T}$ does not prove this fact, as it is for instance not validated by the polynomial algebra $U_\mathbb{T}[X]$.

However, Theorem 3.7 still plays an important role in understanding first-order formulas, as it is at the core of our proposed characterization of the first-order formulas validated by the generic model.

Scholium 3.11. *Let $\mathbb{T}$ be a geometric theory. Let $\vec{x}$ be a context over the signature of $\mathbb{T}$. Let $(\sigma_i)_i$ be a set of geometric sequents, where each context is of the form $\vec{x}, \vec{y}_i$ for some additional list $\vec{y}_i$. Write $\sigma_i = (\varphi_i \vdash_{\vec{x},\vec{y}_i} \psi_i)$. Then, internally to* $\mathrm{Set}[\mathbb{T}]$*, it holds that*

$$\forall \vec{x}.\ \Big(\bigwedge_i (\forall \vec{y}_i.\ \varphi_i \Rightarrow \psi_i)\Big) \Longrightarrow \bigvee_\alpha \alpha,$$

where the disjunction ranges over all those geometric formulas α in the context $\vec{x}$ such that for every index i, the theory $\mathbb{T}$ proves $(\alpha \wedge \varphi_i \vdash_{\vec{x},\vec{y}_i} \psi_i)$.

Proof. Immediate from the proof of Theorem 3.7. □

Theorem 3.12. *Let $\mathbb{T}$ be a geometric theory. Let χ be a (finitary or infinitary) first-order formula over the signature of $\mathbb{T}$. Then the following statements are equivalent.*

(1) *The formula χ holds for $U_\mathbb{T}$.*
(2) *The formula χ is provable in infinitary first-order intuitionistic logic modulo the axioms of $\mathbb{T}$ adjoined by, for each geometric sequent $\sigma = (\varphi \vdash_{\vec{x},\vec{y}} \psi)$ and for each splitting of its context into two parts $\vec{x}, \vec{y}$, the additional axiom*

$$\forall \vec{x}.\ \Big((\forall \vec{y}.\ \varphi \Rightarrow \psi) \Longrightarrow \bigvee_\alpha \alpha\Big), \tag{‡}$$

where the disjunction ranges over all those geometric formulas α in the context $\vec{x}$ such that $\mathbb{T}$ proves $(\alpha \wedge \varphi \vdash_{\vec{x},\vec{y}} \psi)$.

To a first approximation, axiom scheme (‡) is just a rendition of the (nontrivial direction of the) statement of the Nullstellensatz in infinitary first-order logic. However, the conclusion "σ is provable modulo $\underline{\mathbb{T}}/U_\mathbb{T}$" does not seem to be expressible in this logic. The conclusion "$\bigvee_\alpha \alpha$" of axiom scheme (‡) is a strengthening of "σ is provable modulo $\underline{\mathbb{T}}/U_\mathbb{T}$" which can. We recall that in infinitary first-order formulas, we allow set-indexed disjunctions (but not set-indexed conjunctions).

Proof of Theorem 3.12. Any infinitary first-order formula can be simplified to an equivalent geometric formula – on the semantic side by repeatedly applying Scholium 3.11, on the syntactic side by repeatedly applying axiom scheme (‡). Hence we are reduced to the basic fact (Proposition 2.5) that, for geometric formulas φ, $\mathrm{Set}[\mathbb{T}] \models \varphi$ if and only if $\mathbb{T}$ proves φ. □

Remark 3.13. Both in Scholium 3.11 and in the axiom scheme (‡), it suffices to restrict the disjunction "$\bigvee_\alpha \alpha$" to those formulas α which are of the form "$\exists \cdots \exists.\ \varphi$" with φ of Horn type.

The Nullstellensatz of Theorem 3.7 and the characterization of Theorem 3.12 pertain to geometric logic and hence require some care in dealing with the flexibility of disjunctions. In particular, the internal statement of Theorem 3.7 requires external ingredients in referring to geometric* sequents. A natural question is therefore whether, in the case that $\mathbb{T}$ is a coherent theory, the statements of these two theorems can be simplified.

However, while geometric logic is powerful enough to express provability in geometric logic (or rather the strenghtening we employed in axiom scheme (‡)), coherent logic is not powerful enough to express provability in coherent logic. "Coherent logic cannot eat itself." Hence the analogue of Theorem 3.12 cannot be formulated for coherent logic.

A fragment of logic which is still finitary, but is powerful enough to express its own provability predicate, is *coherent existential fixed-point logic*, coherent logic enriched by list sorts and the fixed-point operator of existential fixed-point logic [6, 4]. The list sorts can be used to express raw terms, raw formulas and raw sequents; the fixed-point operator can then be used to express well-formedness of raw terms, raw formulas and raw sequents; and to express provability. For this fragment, we have the following characterization.

Scholium 3.14. *Let $\mathbb{T}$ be a coherent theory (or more generally a theory in coherent existential fixed-point logic). Let α be a finitary first-order formula over the signature of $\mathbb{T}$. Then the following statements are equivalent.*

1. *The formula α holds for $U_\mathbb{T}$.*
2. *The formula α is provable in coherent existential fixed-point logic modulo the axioms of $\mathbb{T}$ and the additional axioms*
$$\ulcorner \sigma \text{ holds for } U_\mathbb{T} \urcorner \Longrightarrow \ulcorner \underline{\mathbb{T}}/U_\mathbb{T} \text{ proves } \sigma \urcorner$$
where σ ranges over the formulas of coherent existential fixed-point logic over the signature of $\mathbb{T}$. (When referring to "$U_\mathbb{T}$", we here mean the tautologous "model" in which any sort of $\mathbb{T}$ is interpreted by the sort itself. For instance, if $\sigma = (\varphi \vdash_{x_1:X_1,\ldots,x_n:X_n} \psi)$ is a sequent, then $\ulcorner \sigma$ holds for $U_\mathbb{T} \urcorner$ is to be interpreted as the formula "$\forall x_1 : X_1. \ldots \forall x_n : X_n.\ (\varphi \Rightarrow \psi)$".)

Proof. The proofs of Theorem 3.7 and Theorem 3.12 can be adapted to the setting of coherent existential fixed-point logic. □

On the other end, we can ask for a generalization of Theorem 3.12 to the extension of infinitary first-order logic where we additionally allow set-indexed conjunctions. Such a generalization is given by the following scholium.

Scholium 3.15. *Let $\mathbb{T}$ be a geometric theory. Let χ be a formula in infinitary first-order formula enriched by set-indexed conjunctions over the signature of $\mathbb{T}$. Then the following statements are equivalent.*

1. *The formula χ holds for $U_\mathbb{T}$.*
2. *The formula χ is provable in infinitary first-order intuitionistic logic enriched by set-indexed conjunctions modulo the axioms of $\mathbb{T}$ adjoined by the displayed formulas of Scholium 3.11.*

Proof. The proof of Theorem 3.12 carries over word for word. □

4. The special case of Horn theories

The purpose of this section is to redo the development in the special case of Horn theories. XXX

Throughout this section, let $\mathbb{T}$ be a Horn theory. It is a basic result that Horn theories are of presheaf type, that is the classifying topos $\mathrm{Set}[\mathbb{T}]$ can be taken as the topos of functors $\mathbb{T}\text{-mod}_{\mathrm{fp}} \to \mathrm{Set}$, where $\mathbb{T}\text{-mod}_{\mathrm{fp}}$ is the full subcategory of the category of $\mathbb{T}$-models (in Set) on the finitely presentable objects [12, Theorem 2.1.21]. The (underlying object of the) generic model is the tautologous functor $U_{\mathbb{T}} : T \mapsto T$. By $U_{\mathbb{T}}$-*algebra* we mean, in analogy with the terminology in the case that $\mathbb{T}$ is the theory of rings, a $\mathbb{T}$-model M equipped with a $\mathbb{T}$-homomorphism $U_{\mathbb{T}} \to M$.

Lemma 4.1. *Let X be a set equipped with a morphism $X \to S$ to the set of sorts of the signature Σ of $\mathbb{T}$. Let R be a set of atomic propositions in which the elements of X may appear as new constants of the respective sorts. Then there is $\mathbb{T}\langle X|R\rangle$, the free $\mathbb{T}$-model on the generators X modulo the relations R.*

Proof. The desired model can be constructed as a term algebra. As a set, it consists of the terms (in the empty context) of the signature $\Sigma + X$ modulo the equivalence relation identifying two terms if and only if $\mathbb{T} + R$ proves them to be equal. The function symbols f of Σ are interpreted by declaring $[\![f]\!]([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$ and the relation symbols H are interpreted by declaring $([t_1], \ldots, [t_n]) \in [\![H]\!] \Leftrightarrow (\mathbb{T} + R \vdash H(t_1, \ldots, t_n))$.

We omit the required verifications and only remark that while the same construction could be carried out if $\mathbb{T}$ was a general geometric theory, the term algebra would in general not be a model of $\mathbb{T}$. □

Lemma 4.2. *Let $\sigma = (\varphi_1 \wedge \cdots \wedge \varphi_n \vdash_{x_1,\ldots,x_k} \psi_1 \wedge \cdots \wedge \psi_m)$ be a Horn sequent over the signature of $\mathbb{T}$. Then the following statements are equivalent.*

(1) *The theory $\mathbb{T}$ proves σ.*
(2) *In $\mathbb{T}\langle x_1, \ldots, x_k \,|\, \varphi_1, \ldots, \varphi_n\rangle$, the propositions $\psi_1, \ldots, \psi_m$ hold for the k-tuple $([x_1], \ldots, [x_k])$.*

Proof. By construction of the term algebra. □

Lemma 4.3. *A $\mathbb{T}$-model is finitely presentable as an object of the category of $\mathbb{T}$-models if and only if it is isomorphic to a model of the form $\mathbb{T}\langle X|R\rangle$ where X is Bishop-finite and R is Kuratowski-finite.*

Proof. It is an instructive exercise to verify that models of the stated form are compact (in a slightly different context, this is done in [1, Theorem 3.12]). Conversely, let a $\mathbb{T}$-model M be given. Then $\mathbb{T}$ is the filtered colimit of all models over M which are of the stated form. If M is compact, the identity on M factors over such a model. Hence M is a retract of such a model and hence itself isomorphic to a model of this form. □

Lemma 4.4. *The category of $\mathbb{T}$-models (in* Set*) is complete and cocomplete.*

Proof. Limits are computed as the limits of the underlying sets, colimits are computed by using the construction of Lemma 4.1. For instance, the coproduct of $\mathbb{T}\langle X|R\rangle$ and $\mathbb{T}\langle X'|R'\rangle$ is $\mathbb{T}\langle X \amalg X' \,|\, R \cup R'\rangle$. □

Having the special case of the theory of rings in mind, we write the coproduct in the category of $\mathbb{T}$-models as "$\otimes$".

Any $\mathbb{T}$-model A has a mirror image in the topos $\mathrm{Set}[\mathbb{T}]$, namely the functor $A^\sim$: $\mathbb{T}\text{-mod}_{\mathrm{fp}} \to \mathrm{Set}$ given by $T \mapsto A \otimes T$. This object is in a canonical way a $\mathbb{T}$-model over $U_\mathbb{T}$, hence from the point of view of $\mathrm{Set}[\mathbb{T}]$ a $\underline{\mathbb{T}}/U_\mathbb{T}$-model.

Lemma 4.5. *The functor $(\cdot)^\sim$ from $\mathbb{T}$-models to $\underline{\mathbb{T}}/U_\mathbb{T}$-models in* $\mathrm{Set}[\mathbb{T}]$ *is left adjoint to the functor $\Gamma = \mathrm{Hom}(1, \cdot)$ computing global elements.*

Proof. A $U_\mathbb{T}$-algebra homomorphism $\alpha : A^\sim \to M$ yields the $\mathbb{T}$-model homomorphism $\alpha_0 : A \to M(0) = \Gamma(M)$, where 0 is the initial $\mathbb{T}$-model. Conversely, a $\mathbb{T}$-model homomorphism $\beta : A \to \Gamma(M)$ yields a $U_\mathbb{T}$-algebra homomorphism by summing $A \to M(0) \to M(T)$ with the structure morphism $T = U_\mathbb{T}(T) \to M(T)$. $\square$

Definition 4.6. The *spectrum* $\mathrm{Spec}(M)$ of a $U_\mathbb{T}$-algebra M in $\mathrm{Set}[\mathbb{T}]$ is the result of constructing, internally to $\mathrm{Set}[\mathbb{T}]$, the set of $U_\mathbb{T}$-algebra homomorphisms $M \to U_\mathbb{T}$.

Lemma 4.7. *Let A be a $\mathbb{T}$-model (not necessarily finitely presentable). Then* $\mathrm{Spec}(A^\sim)$ *coincides with $\text{よ}A$, the functor* $\mathrm{Hom}_{\mathbb{T}\text{-mod}}(A, \cdot)$.

Proof. By the Yoneda lemma, the sections of the presheaf $\mathrm{Spec}(A^\sim) : \mathbb{T}\text{-mod}_{\mathrm{fp}} \to \mathrm{Set}$ on an object T are given by the set

$$\begin{aligned}
\mathrm{Spec}(A^\sim)(T) &\cong \mathrm{Hom}(\text{よ}T, \mathrm{Spec}(A^\sim)) = \mathrm{Hom}(\text{よ}T, [A^\sim, U_\mathbb{T}]_{U_\mathbb{T}}) \\
&\cong \mathrm{Hom}(\text{よ}T \times A^\sim, U_\mathbb{T})_{U_\mathbb{T}\text{-algebra homomorphism in second argument}} \\
&\cong \mathrm{Hom}_{U_\mathbb{T}}(A^\sim, (U_\mathbb{T})^{\text{よ}T}) \cong \mathrm{Hom}_{U_\mathbb{T}}(A^\sim, U_\mathbb{T}|T),
\end{aligned}$$

where $[A^\sim, U_\mathbb{T}]_{U_\mathbb{T}}$ is the object of $U_\mathbb{T}$-algebra homomorphisms from $A^\sim$ to $U_\mathbb{T}$ (a subobject of the internal Hom $(U_\mathbb{T})^{A^\sim}$); $\mathrm{Hom}_{U_\mathbb{T}}$ denotes the set of $U_\mathbb{T}$-algebra homomorphisms; $(U_\mathbb{T})^{\text{よ}T}$ is the object of morphisms from $\text{よ}T$ to $U_\mathbb{T}$; and $U_\mathbb{T}|T$ is the functor $U_\mathbb{T}(T \times \cdot)$, that is the functor $S \mapsto T \otimes S$.

An arbitrary element $f \in (\text{よ}A)(T)$, that is an arbitrary $\mathbb{T}$-model homomorphism $f : A \to T$, induces a $U_\mathbb{T}$-algebra homomorphism $g : A^\sim \to U_\mathbb{T}|T$ by setting $g_S := f \otimes \mathrm{id}_S : A \otimes S \to T \otimes S$. The given homomorphism f can be recovered by $f = g_0$, the component of g at the initial model.

Conversely, a $U_\mathbb{T}$-algebra homomorphism $g : A^\sim \to U_\mathbb{T}|T$ induces a $\mathbb{T}$-model homomorphism $f : A \to T$ by setting $f := g_0$. Because g is a natural transformation and because g is compatible with the structure morphisms $U_\mathbb{T} \to A^\sim$ and $U_\mathbb{T} \to U_\mathbb{T}|T$, the morphism g is determined by f. $\square$

Lemma 4.8. *Let A be a finitely presentable $\mathbb{T}$-model. Then the canonical morphism*

$$A^\sim \longrightarrow (U_\mathbb{T})^{\mathrm{Spec}(A^\sim)}$$

is an isomorphism of $U_\mathbb{T}$-algebras.

Proof. By Lemma 4.7, the functor $\mathrm{Spec}(A^\sim)$ coincides with $\text{よ}A$. Since by assumption A is contained in the site defining $\mathrm{Set}[\mathbb{T}]$, the exponential $(U_\mathbb{T})^{\text{よ}A}$ coincides with $U_\mathbb{T}|A$ (notation as in the proof in Lemma 4.7), that is, with the $U_\mathbb{T}$-algebra $A^\sim$. $\square$

Corollary 4.9. *Let A and B be $\mathbb{T}$-models. Assume that B is finitely presentable. Then the canonical morphism*

$$\mathrm{Hom}_{U_\mathbb{T}}(A^\sim, B^\sim) \longrightarrow \mathrm{Spec}(A^\sim)^{\mathrm{Spec}(B^\sim)}$$

is an isomorphism.

Proof. We have the chain of isomorphisms

$$\begin{aligned}\mathrm{Spec}(A^\sim)^{\mathrm{Spec}(B^\sim)} = ([A^\sim, U_{\mathbb{T}}]_{U_{\mathbb{T}}})^{\mathrm{Spec}(B^\sim)} &\cong [\mathrm{Spec}(B^\sim) \times A^\sim, U_{\mathbb{T}}]_{U_{\mathbb{T}}} \\ &\cong [A^\sim, U_{\mathbb{T}}^{\mathrm{Spec}(B^\sim)}]_{U_{\mathbb{T}}} \cong [A^\sim, B^\sim],\end{aligned}$$

where the final isomorphism is by Lemma 4.8. □

Theorem 4.10. *The generic* $\mathbb{T}$*-model is* quasicoherent *in the following sense: From the point of view of* $\mathrm{Set}[\mathbb{T}]$*, for any finitely presentable* $U_{\mathbb{T}}$*-algebra* A *(finitely presentable object in the category of* $U_{\mathbb{T}}$*-algebras), the canonical* $U_{\mathbb{T}}$*-algebra homomorphism*

$$A \longrightarrow (U_{\mathbb{T}})^{\mathrm{Spec}(A)},\ x \longmapsto -(x)$$

is an isomorphism.

We use the term *quasicoherent* in reference to algebraic geometry: For an $\underline{\mathbb{A}}^1$-module M in the big Zariski topos of a scheme S, where $\underline{\mathbb{A}}^1$ is the functor of points of the affine line over S, there is a well-established notion of what it means that M is quasicoherent. This property can be characterized in the internal language: Such a module M is quasicoherent if and only if, from the internal point of view of the big Zariski topos, the canonical map $A \otimes_{\underline{\mathbb{A}}^1} M \to M^{\mathrm{Spec}(A)}$ is an isomorphism for any finitely presented $\underline{\mathbb{A}}^1$-algebra A [9, Theorem 18.19]. Specializing to the case $M = \underline{\mathbb{A}}^1$, we obtain the quasicoherence condition of Theorem 4.10.

Proof of Theorem 4.10. The proof of Lemma 4.3 is constructive and thus valid in the internal language of $\mathrm{Set}[\mathbb{T}]$. Hence we can apply it, internally, to the theory $\underline{\mathbb{T}}/U_{\mathbb{T}}$ to deduce that a $U_{\mathbb{T}}$-algebra A is finitely presentable if and only if it is isomorphic to a $U_{\mathbb{T}}$-algebra of the form $(\underline{\mathbb{T}}/U_{\mathbb{T}})\langle X|R\rangle$ with X Bishop-finite and R Kuratowski-finite.

We therefore have to verify the following internal statement: *For any number* n*, for any sorts* $X_1, \ldots, X_n$ *of* $\underline{\mathbb{T}}/U_{\mathbb{T}}$*, for any number* m*, for any atomic propositions* $R_1, \ldots, R_m$ *over the signature of* $\underline{\mathbb{T}}/U_{\mathbb{T}}$ *extended by constants* $e_1 : X_1, \ldots, e_n : X_n$*, the canonical map* $A \to (U_{\mathbb{T}})^{\mathrm{Spec}(A)}$ *where* $A := (\underline{\mathbb{T}}/U_{\mathbb{T}})\langle e_1 : X_1, \ldots, e_n : X_n \mid R_1, \ldots, R_m\rangle$ *is an isomorphism.*

Following the Kripke–Joyal translation of this statement, let a stage $T \in \mathbb{T}\text{-mod}_{\mathrm{fp}}$, T-elements $X_1, \ldots, X_n$ of the object of sorts of the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$ (that is the constant presheaf on the set of sorts of $\mathbb{T}$), and T-elements $R_1, \ldots, R_m$ of the object of atomic propositions over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$ be given. The X_i are given by sorts of $\mathbb{T}$ and the R_j are given by atomic propositions over the signature of $\mathbb{T}/U_{\mathbb{T}}(T)$.

Since the slice $\mathrm{Set}[\mathbb{T}]/ {\text{よ}} T$ is equivalent to $\mathrm{Set}[\mathbb{T}/T]$, hence again the classifying topos of a Horn theory, we may without loss of generality assume that T is the initial $\mathbb{T}$-model.

In this case the claim follows from Lemma 4.8, since the result of constructing, internally to $\mathrm{Set}[\mathbb{T}]$, the model $(\underline{\mathbb{T}}/U_{\mathbb{T}})\langle e_1 : X_1, \ldots, e_n : X_n \mid R_1, \ldots, R_m\rangle$ coincides with the $U_{\mathbb{T}}$-algebra $(\mathbb{T}\langle e_1 : X_1, \ldots, e_n : X_n \mid R_1, \ldots, R_m\rangle)^\sim$. □

Theorem 4.11. *Let* $\mathbb{T}'$ *be a geometric quotient theory of* $\mathbb{T}$*, not necessarily Horn. Assume that* $U_{\mathbb{T}}$ *is a sheaf for the topology on* $\mathrm{Set}[\mathbb{T}]$ *cutting out the subtopos* $\mathrm{Set}[\mathbb{T}']$*. Then, from the internal point of view of* $\mathrm{Set}[\mathbb{T}']$*, the canonical map* $A \to (U_{\mathbb{T}'})^{\mathrm{Spec}(A)}$ *is an isomorphism of* $U_{\mathbb{T}}$*-algebras for every finitely* *presented $U_{\mathbb{T}}$-algebra A. (By "Spec(A)", we here mean the set of $U_{\mathbb{T}}$-algebra homomorphisms from A to $U_{\mathbb{T}'}$.)*

Proof. In general, the generic model $U_{\mathbb{T}'}$ is the sheafification of $U_{\mathbb{T}}$ [13, Lemma 2.3]. By the sheaf assumption, the objects $U_{\mathbb{T}'}$ and $U_{\mathbb{T}}$ actually agree as functors on $\mathbb{T}\text{-mod}_{\mathrm{fp}}$.

Following the Kripke–Joyal translation of the claim, similar as in the proof of Theorem 4.10, it suffices to show that applying the sheafification functor to the isomorphisms provided by Lemma 4.8 (over arbitrary slices) results in isomorphisms of the same form. This fact follows from the fact that the presheaf Hom into a sheaf is already a sheaf itself and from the observation that the construction "free $U_{\mathbb{T}}$-module modulo relations" is geometric, hence in particular commutes with sheafification. ☐

Remark 4.12. The finite presentability condition in Lemma 4.8 cannot be dropped. For instance, in the case that $\mathbb{T}$ is the theory of commutative rings with unit and A is the ring $\mathbb{Q}$ of rational numbers, we have $\mathrm{Spec}(A^\sim) \cong \mathrm{Spec}(0^\sim)$, where 0 is the zero ring, as $\mathbb{Q}$ allows ring homomorphisms only to those finitely presented rings in which $1 = 0$ holds. Hence $A^\sim$ and $(U_{\mathbb{T}})^{\mathrm{Spec}(A^\sim)} \cong (U_{\mathbb{T}})^{\mathrm{Spec}(0^\sim)} \cong 0^\sim$ do not coincide.

5. The generalization to the higher-order case

By *extended geometric logic* we mean the extension of geometric logic where we are allowed to form, in addition to the basic sorts supplied by a given signature, finite limits of sorts and set-indexed colimits of sorts. By (intuitionistic) *higher-order logic*, we mean the further extension where we may also form powersorts. These derived sorts come with respective term constructors (tuple formers, coprojections, set comprehension) and the usual rules governing these constructors.

An *extended geometric formula* is a formula of extended geometric logic built from equality and relation symbols by the logical connectives $\top \perp \wedge \vee \exists$ and by arbitrary set-indexed disjunctions $\bigvee$. Existential quantification can be over any of the sorts of extended geometric logic, including the derived sorts. An *extended geometric sequent* is a sequent of the form $(\varphi \vdash_{\vec{x}} \psi)$ where φ and ψ are extended geometric formulas and the sorts of the variables $\vec{x}$ may be derived sorts.

It is possible to extend the Kripke–Joyal semantics so that higher-order logic can be interpreted in any Grothendieck topos. The truth of a higher-order sequent $(\varphi \vdash_{\vec{x}} \psi)$ is in general not preserved under pullback along geometric morphisms, even if φ and ψ do not contain $\forall$ and $\Rightarrow$, since powerobjects are in general not preserved under pullback. However, as can be deduced from the following lemma, the truth of extended geometric sequents is preserved; as is folklore, extended geometric logic is just a thin layer over ordinary geometric logic.

Lemma 5.1. *Let σ be an extended geometric sequent over the signature of a geometric theory $\mathbb{T}$. Then there is a set-indexed family $(\sigma_i)_i$ of ordinary geometric sequents over the same signature such that σ is provable in extended geometric logic if and only if all the sequents σ_i are provable in ordinary geometric logic.*

Proof. Any existential quantification of the form "$\exists p : X \times Y$" can be replaced by the string "$\exists x : X.\ \exists y : Y$", and similarly for free variables of product sorts appearing in the context of σ. In a similar vein more general finite limits are treated.

An existential quantification of the form "$\exists x : \coprod_i X_i$" can be replaced by the string "$\bigvee_i \exists x : X_i$".

Finally, for any occurrence of a free variable $x\colon \coprod_i X_i$ in the context of σ, we can replace σ by the family of sequents $(\sigma_i)_i$, where the sequent σ_i is the same as σ only that the free variable x is changed to be of sort X_i (and the corresponding change in the consequent and the antecedent is applied, applying the appropriate coprojection).

After carrying out these steps, the free variables are only of the basic sorts supplied by the signature of $\mathbb{T}$ and existential quantifications only range over the basic sorts. However, in the consequents and antecedents, still tuple formers and coprojections may appear. These can be replaced as suggested by the rules governing these. For instance, an equation "$\langle x, y\rangle = \langle x', y'\rangle$" can be replaced by the conjunction "$x = x' \wedge y = y'$", and an equation "$\iota_i(x) = \iota_j(y)$" (where ι_i and ι_j are coprojections associated with coproduct sorts) can be replaced by the subsingleton-indexed disjunction "$\bigvee\{x = y \mid i = j\}$". ☐

Theorem 5.2. *Let $\mathbb{T}$ be a geometric theory. Let $x_1 : X_1, \ldots, x_n : X_n$ be a context over the signature of $\mathbb{T}$. Let $\mathrm{Form}^\star_{\vec{x}}(\underline{\mathbb{T}}/U_{\mathbb{T}})/(\dashv\vdash_{\vec{x}})$ be the $\mathrm{Set}[\mathbb{T}]$-object of geometric* formulas over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$ in the context $\vec{x}$, where any two such formulas are identified if and only if $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves them equivalent. Then the canonical morphism*

$$\mathrm{Form}^\star_{\vec{x}}(\underline{\mathbb{T}}/U_{\mathbb{T}})/(\dashv\vdash_{\vec{x}}) \longrightarrow P(\text{よ}X_1 \times \cdots \times \text{よ}X_n)$$

sending, internally speaking, the equivalence class of a geometric* formula φ to the subset $\{(x_1, \ldots, x_n) \mid \varphi \text{ holds for } (x_1, \ldots, x_n)\}$ is an isomorphism.

Proof. Injectivity is by the Nullstellensatz of Theorem 3.7. Surjectivity is by the definability result [13, Theorem 2.4], exploiting that the internal statement localizes well by Lemma 3.5. ☐

Remark 5.3. The proof of Theorem 5.2 used the Nullstellensatz of Theorem 3.7. Conversely, Theorem 5.2 implies Theorem 3.7. Indeed, arguing internally, let $\sigma = (\varphi \vdash_{\vec{x}} \psi)$ be a geometric* sequent over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$ such that σ holds for $U_{\mathbb{T}}$. Then the subsets $\{(\vec{x}) \mid \varphi\}$ and $\{(\vec{x}) \mid \varphi \wedge \psi\}$ are equal. Hence the assumption implies that the formulas φ and $\varphi \wedge \psi$ are provably equivalent. Thus $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves $(\varphi \vdash_{\vec{x}} \psi)$.

Corollary 5.4. *Let $\mathbb{T}$ be a geometric theory. Let $\{\vec{x}.\ \varphi\}$ and $\{\vec{y}.\ \psi\}$ be geometric formulas in given contexts. Then, internally to $\mathrm{Set}[\mathbb{T}]$, the canonical map from the set of equivalence classes of $\underline{\mathbb{T}}/U$-provably functional geometric* formulas from $\{\vec{x}.\ \varphi\}$ to $\{\vec{y}.\ \psi\}$ to the set of maps $\{(\vec{y}) \mid \psi\}^{\{(\vec{x}) \mid \varphi\}}$ is a bijection.*

Proof. We argue internally to $\mathrm{Set}[\mathbb{T}]$. The canonical map sends an equivalence class $[\theta]$ to the unique map $f : \{(\vec{x}) \mid \varphi\} \to \{(\vec{y}) \mid \psi\}$ whose graph is given by the set $\{(\vec{x}, \vec{y}) \mid \theta\}$.

For verifying surjectivity, let a map $f : \{(\vec{x}) \mid \varphi\} \to \{(\vec{y}) \mid \psi\}$ be given. Then its graph is a subset of $\vec{X} \times \vec{Y}$, hence by Theorem 5.2 given by a geometric* formula θ. Because f is a map, this formula is functional; and by the Nullstellensatz, it is $\underline{\mathbb{T}}/U_{\mathbb{T}}$-provably so.

For verifying injectivity, let θ and θ' be $\underline{\mathbb{T}}/U_{\mathbb{T}}$-provably functional formulas which give rise to identical maps. Then they also give rise to identical graphs, hence are $\underline{\mathbb{T}}/U_{\mathbb{T}}$-provably equivalent by Theorem 5.2. ☐

Theorem 5.5. *Let $\mathbb{T}$ be a geometric theory. Then, internally to $\mathrm{Set}[\mathbb{T}]$, for any extended geometric* sequent σ over the signature of $\underline{\mathbb{T}}/U$, the following statements are equivalent:*

(1) *The sequent σ holds for $U_{\mathbb{T}}$.*

(2) *The sequent σ is provable modulo $\underline{\mathbb{T}}/U_{\mathbb{T}}$ in extended geometric logic.*

Proof. The implication (2) $\Rightarrow$ (1) is immediate since $U_{\mathbb{T}}$ is a model of $\underline{\mathbb{T}}/U_{\mathbb{T}}$. The converse direction is by Lemma 5.1, which holds internally in $\mathrm{Set}[\mathbb{T}]$ as the proof we supplied is constructive, and by the Nullstellensatz for ordinary geometric logic of Theorem 3.7. $\square$

Theorem 5.6. *Let $\mathbb{T}$ be a geometric theory. Let χ be a higher-order formula over the signature of $\mathbb{T}$. Then the following statements are equivalent:*

(1) *The formula χ holds for $U_{\mathbb{T}}$.*

(2) *The formula χ is provable in higher-order intuitionistic logic modulo the axioms of $\mathbb{T}$ and the additional axiom scheme*

$$\ulcorner\text{the map } \mathrm{Form}^{\star}_{\vec{x}}(\underline{\mathbb{T}}/U_{\mathbb{T}})/(\dashv\vdash_{\vec{x}}) \longrightarrow P(X_1 \times \cdots \times X_n) \text{ is bijective}\urcorner, \qquad (\P)$$

where $X_1, \ldots, X_n$ is any list of sorts.

Proof. Theorem 5.2 on the semantic side and the axiom scheme (¶) on the syntactic side allow us to replace any mention of a powersort $P(X)$ in χ by $\mathrm{Form}^{\star}_{x:X}(\underline{\mathbb{T}}/U_{\mathbb{T}})/(\dashv\vdash_x)$, similarly to how the proof of Lemma 5.1 compiles extended geometric logic to ordinary geometric. Then we can argue as in the proof of Theorem 3.12, noting that the axiom scheme indeed entails the axiom scheme (‡) by Remark 5.3. $\square$

6. Applications

6.1. **Nongeometric sequents in the object classifier.** Butz and Johnstone [11] list several nongeometric sequents holding in the classifying topos of the theory $\mathbb{O}$ of objects, the theory with a single sort and no function symbols, relation symbols or axioms. In particular, they present the following ones.

$$\forall x, y : U_{\mathbb{O}}.\ \neg\neg(x = y) \qquad\qquad \forall x_1, \ldots, x_n : U_{\mathbb{O}}.\ \neg\forall y : U_{\mathbb{O}}.\ \bigvee_{i=1}^{n} (y = x_i)$$

Intuitively, the first sequent expresses that $U_{\mathbb{O}}$ is close to a subsingleton, while the second expresses that $U_{\mathbb{O}}$ is close to being infinite. Using the Nullstellensatz, these formulas can be verified as follows.

We argue internally to $\mathrm{Set}[\mathbb{O}]$. Let $x, y : U_{\mathbb{O}}$ and assume $\neg(x = y)$. By the Nullstellensatz, the theory $\underline{\mathbb{O}}/U_{\mathbb{O}}$ proves $(x = y \vdash_{[]} \bot)$. Hence this sequent holds of any model of $\underline{\mathbb{O}}/U_{\mathbb{O}}$, in particular of the quotient $U_{\mathbb{O}}/(x \sim y)$. Hence $\bot$ holds for this model, that is, we have a contradiction.

For the second formula, let $x_1, \ldots, x_n : U_{\mathbb{O}}$ and assume $\forall y : U_{\mathbb{O}}.\ \bigvee_{i=1}^{n}(y = x_i)$. By the Nullstellensatz, this universal statement is provable and hence holds for every $\underline{\mathbb{O}}/U_{\mathbb{O}}$-model. But it does not hold for $U_{\mathbb{O}} \amalg \{\star\}$.

The argument using the Nullstellensatz makes it transparent that both formulas are also satisfied by the generic model of any Horn theory $\mathbb{T}$: In that case, instead of constructing the quotient set $U_{\mathbb{O}}/(x \sim y)$, we construct the quotient model $(\underline{\mathbb{T}}/U_{\mathbb{T}})\langle\,|\, x = y\rangle$; and instead of $U_{\mathbb{O}} \amalg \{\star\}$, we construct the free model $(\underline{\mathbb{T}}/U_{\mathbb{T}})\langle \star \,|\, \rangle$ (XXX need to exclude trivial singleton theory).

6.2. **Nongeometric sequents in the ring classifier.** Each of the nongeometric sequents mentioned in Section 1 can be deduced from the first-order or higher-order Nullstellensatz. Here we briefly indicate how this works using two examples.

Let $\mathbb{T}$ be the theory of rings. Then, internally to $\mathrm{Set}[\mathbb{T}]$, we have for any element $x : U_{\mathbb{T}}$:

$$(x = 0 \Rightarrow 1 = 0) \Longrightarrow \ulcorner x \text{ is invertible} \urcorner \quad \text{and}$$
$$(\ulcorner x \text{ is invertible} \urcorner \Rightarrow 1 = 0) \Longrightarrow \ulcorner x \text{ is nilpotent} \urcorner.$$

To verify the first claim, assume $(x = 0 \Rightarrow 1 = 0)$. By the Nullstellensatz of Theorem 3.7, the theory $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves this fact. Hence in particular it holds in the $U_{\mathbb{T}}$-algebra $U_{\mathbb{T}}/(x)$. Because $x = 0$ in $U_{\mathbb{T}}/(x)$, we have $1 = 0$ in $U_{\mathbb{T}}/(x)$, that is $1 \in (x)$, hence x is invertible.

To verify the second claim, assume $(\ulcorner x \text{ is invertible} \urcorner \Rightarrow 1 = 0)$. By the Nullstellensatz, the theory $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves this fact, hence in particular it holds in $U_{\mathbb{T}}[x^{-1}]$. Because x is invertible in this $U_{\mathbb{T}}$-algebra, we have $1 = 0$ in $U_{\mathbb{T}}[x^{-1}]$. Hence x is nilpotent.

Thanks to the version of the Nullstellensatz pertaining Horn consequents given in Corollary 3.9, these two arguments can also be carried out in the classifying topos of local rings. It is instructive to consider how we would have to argue if we wanted to use only the unmodified Nullstellensatz:

Assume $(x = 0 \Rightarrow 1 = 0)$. By the Nullstellensatz, the theory of local rings proves this fact. From this we may not conclude that it holds for $U_{\mathbb{T}}/(x)$, as this $U_{\mathbb{T}}$-algebra might not be local. But we may conclude that it holds for $\mathcal{O}_{\mathrm{Spec}(U_{\mathbb{T}}/(x))}$, the structure sheaf of the affine scheme given by $U_{\mathbb{T}}/(x)$. This sheaf is not a model of $\underline{\mathbb{T}}/U_{\mathbb{T}}$, but it is so from the point of view of the topos of sheaves over $\mathrm{Spec}(U_{\mathbb{T}}/(x))$. From the point of view of that topos, $x = 0$ in $\mathcal{O}_{\mathrm{Spec}(U_{\mathbb{T}}/(x))}$, hence $1 = 0$ in $\mathcal{O}_{\mathrm{Spec}(U_{\mathbb{T}}/(x))}$. This amounts to $U_{\mathbb{T}}/(x)$ being the zero ring, hence x is invertible in $U_{\mathbb{T}}$.

6.3. **On the Kock–Lawvere axiom in synthetic differential geometry.** The generic local ring $\underline{\mathbb{A}}^1$ validates the following strong form of the Kock–Lawvere axiom (Theorem 4.10): *For any finitely presented $\underline{\mathbb{A}}^1$-algebra A, the canonical map $A \to (\underline{\mathbb{A}}^1)^{\mathrm{Spec}(A)}$ is an isomorphism of $\underline{\mathbb{A}}^1$-algebras.* The special case $A := \underline{\mathbb{A}}^1[X]$ implies that any map $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is given by a polynomial, since $\mathrm{Spec}(\underline{\mathbb{A}}^1[X]) = \mathrm{Hom}_{\underline{\mathbb{A}}^1}(\underline{\mathbb{A}}^1[X], \underline{\mathbb{A}}^1) \cong \underline{\mathbb{A}}^1$.

In contrast, in most models of synthetic differential geometry, we only have the following weaker statement: *For any Weil algebra A, the canonical map $A \to \mathbb{R}^{\mathrm{Spec}(A)}$ is an isomorphism of $\mathbb{R}$-algebras.* Here $\mathbb{R}$ refers to the canonical line object (the image of the manifold $\mathbb{R}^1$ in the model) and $\mathrm{Spec}(A)$ is the set of $\mathbb{R}$-algebra homomorphisms $A \to \mathbb{R}$. Any Weil algebra is finitely presented, but the converse fails; in particular, the polynomial algebra $\mathbb{R}[X]$ is not a Weil algebra, hence there is no reason to believe that any map $\mathbb{R} \to \mathbb{R}$ is given by a polynomial.

The quasicoherence statement of Theorem 4.10 can shed light on this discrepancy.

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