HIGHER DIRECT IMAGES FOR DUMMIES

INGO BLECHSCHMIDT, PASCUALE ZENOBIO DE ROSSI

Abstract. Your abstract.

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2. Basics on sheaves

2.1. **Presheaves and sheaves.**

Definition 2.1. A *presheaf* $\mathcal{E}$ on a topological space X is a functor $\mathrm{Ouv}(X)^{\mathrm{op}} \to \mathrm{Set}$, where $\mathrm{Ouv}(X)$ is the category of open subsets of X. For an open subset U, the elements of $\mathcal{E}(U)$ are called *sections of $\mathcal{E}$ over U*.

Definition 2.2. A *sheaf* is a presheaf $\mathcal{E}$ which satisfies, for any open covering $U = \bigcup_i U_i$ of any open subset U, the following *glueing condition*: For any family $(s_i)_i$ of sections $s_i \in \mathcal{E}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, there exists a unique section $s \in \mathcal{E}(U)$ such that $s|_{U_i} = s_i$ for all i.

Remark 2.3. Let $\mathcal{E}$ be a sheaf. Then $\mathcal{E}(\emptyset)$ is a singleton set, by the glueing condition for the trivial open covering of the empty set. In contrast, a presheaf may take arbitrary values on the empty set.

Example 2.4. On any topological space X, there is the sheaf $\mathcal{C}$ (or $\mathcal{C}_{\mathbb{R}}$) of real-valued functions. On an open subset U, its sections are the continuous functions $U \to \mathbb{R}$.

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Example 2.5. On any manifold X, there is the sheaf $\mathcal{C}^\infty$ of smooth real-valued functions.

Example 2.6. Let A be a set. Then, on any topological space X, there is the so called *constant presheaf* with $U \mapsto A$ (and identities as restriction maps). For nonempty sets A, this presheaf is only on a sheaf in the case that X is the empty space. There is also the *constant sheaf* $\underline{A}$, which has sections over an open set U the locally constant functions $U \to A$ (equivalently, the continuous functions $U \to A$, where A is equipped with the discrete topology).

Example 2.7. Let $f : E \to X$ be a continuous map. Think of f as a (not necessarily locally trivial) bundle, i. e. visualize not the operation $e \mapsto f(e)$, but the fibers $f^{-1}[x]$. Then there is the *sheaf of sections of* f. Its sections over an open subset U are continuous functions $s : U \to E$ with $f \circ s = \mathrm{id}|_U$.

2.2. **Stalks.** We picture a sheaf not by looking at its vast collection of sets of sections (which are indexed by the open subsets of the base space), but by looking at its *stalks*. There are indexed by the much smaller and much more geometric set of the points of the base space.

Definition 2.8. Let $\mathcal{E}$ be a presheaf of a topological space X. Let $x \in X$. The *stalk* of $\mathcal{E}$ at x, denoted $\mathcal{E}_x$, is the set of germs of sections of $\mathcal{E}$ defined on an open neighbourhood of x. More formally, it is given by the formula

$$\mathcal{E}_x := \mathrm{colim}_{x \in U \subseteq X} \mathcal{E}(U) = \{\langle U, s \in \mathcal{E}(U)\rangle \mid x \in U, U \subseteq X \text{ open}\}/\sim,$$

where sections $\langle U, s\rangle$ and $\langle V, t\rangle$ are identified if and only if they agree of an open neighbourhood of x in $U \cap V$. The image of a section s in $\mathcal{E}_x$ is also denoted s_x.

Example 2.9. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on the complex plane $X = \mathbb{C}$. Then the stalk of $\mathcal{O}_X$ at the origin, $\mathcal{O}_{X,0}$, is isomorphic to the set of power series with positive convergence radius.

Warning 2.10. Let $\pi : E \to X$ be a vector bundle. Let $\mathcal{E}$ be the associated sheaf of sections (example 2.7). Then, in most situations, the stalks $\mathcal{E}_x$ do *not* agree with the fibers $E_x = \pi^{-1}[x]$. This is because $\mathcal{E}_x$ contains *germs* of sections while E_x only contains the *values* of those sections at x. The precise relation is $E_x \cong \mathcal{E}_x \otimes_{\mathcal{C}_x} \mathbb{R}$, where $\mathbb{R}$ is a $\mathcal{C}_x$-module by the scalar multiplication $f \cdot a := f(x)a$.

Proposition 2.11. *Let $\mathcal{E}$ be a sheaf on a topological space X. Let s and t be sections over some open subset U. Then $s = t$ if and only if $s_x = t_x$ for all $x \in U$.*

2.3. **Sections on arbitrary subsets.**

Definition 2.12. Let $\mathcal{E}$ be a sheaf on a topological space. Let A be an arbitrary subset of X, not necessarily open. Then $\Gamma(A, \mathcal{E})$, the *set of sections of $\mathcal{E}$ over A*, is defined by the formula

$$\Gamma(A, \mathcal{E}) := \mathrm{colim}_{A \subseteq U \subseteq X} \mathcal{E}(U),$$

where the colimit ranges over all open subsets U of X containing A. In explicit terms, an element of $\Gamma(A, \mathcal{E})$ is represented by a section on some open neighbourhood of A, where two such sections are identified if and only if they agree on an open neighbourhood of A in their common domain.

Example 2.13. Let A consist of n distinct discrete points $x_1, \dots, x_n$. Then $\Gamma(A, \mathcal{E}) \cong \mathcal{E}_{x_1} \oplus \cdots \oplus \mathcal{E}_{x_n}$. **XXX** proof.

XXX: Example with the sheaves $\mathcal{C}$ and $\mathcal{C}^1$.

2.4. **Sheafification.**

2.5. **Exact sequences.**

2.6. **Inverse images.**

3. Locally constant sheaves

- Give definition.
- Explain sources of examples (and their usefulness): constant sheaves (duh), direct images (sometimes), sheaves of solutions of differential equations; and of course examples induced by covering spaces, representations of the fundamental groupoid, and vector bundles with flat connection.
- Explain the equivalence of locally constant sheaves and covering spaces; also discuss how f^* and f_* look in this picture.
- Explain the equivalence of locally constant sheaves and representations of the fundamental groupoid. Calculate examples of the induced representation, for instance with the sheaf $\{f \mid f' = \frac{1}{2z} f\}$ on $\mathbb{C}^\times$ (which has monodromy $f \mapsto -f$). Also explain how to restrict on representations of the fundamental group at some fixed base point (find out the technical hypotheses needed for this).
- Explain the equivalence of locally constant sheaves and vector bundles with flat connection. Warn that a visual representation of a vector bundles necessarily lives in $\mathbb{R}^3$ and thus, somewhat unfortunately, comes with an induced connection.

4. Direct images

Definition 4.1. Let $f : X \to Y$ be a continuous map. Let $\mathcal{E}$ be a sheaf on X. Then the *direct image of $\mathcal{E}$ along f* is the sheaf $f_*\mathcal{E}$ on Y given by $(f_*\mathcal{E})(V) := \mathcal{E}(f^{-1}[V])$.

Proposition 4.2. *Let $f : X \to Y$ be a closed continuous map. Let $\mathcal{E}$ be a sheaf on X. Then the stalks of the direct image are given by $(f_*\mathcal{E})_y \cong \Gamma(f^{-1}[y], \mathcal{E})$.*

Proof. Recall, for instance from [7, Lemma 8.10], that a continuous map $f : X \to Y$ is closed if and only if for all $y \in Y$ and all open neighbourhoods U of $f^{-1}[y]$ in X there exists an open neighbourhood V of y such that $f^{-1}[V] \subseteq U$. Then compare the systems of which the colimit is taken:

- For calculating $(f_*\mathcal{E})_y$, the colimit is over the sets $\mathcal{E}(f^{-1}[V])$, where V ranges over the open neighbourhoods of y.
- For calculating $\Gamma(f^{-1}[y], \mathcal{E})$, the colimit is over the sets $\mathcal{E}(U)$, where U ranges oven the open neighbourhoods of $f^{-1}[y]$.

Any set which appears in the first system also appears in the second system. Conversely, by the closedness assumption, any set of the second system can be refined to a set of the first system. In more formal terms, the first system is included cofinally into the second. Therefore the colimits agree. □

Example 4.3. In general, the statement is false if the closedness condition is dropped. For instance, consider the inclusion $j : \mathbb{R}\setminus\{0\} \to \mathbb{R}$. Then $(j_*\mathbb{Z})_0 \cong \mathbb{Z}^2$, but $\Gamma(j^{-1}[0], \mathbb{Z}) = 0$.

- Explain why closedness is intuitively necessary.
- Give description in terms of generators and relations (in the case that this should turn out to be helpful).

5. Sheaf cohomology and higher direct images

5.1. **Basic definition and properties.**

Definition 5.1. Let $\mathcal{E}$ be a sheaf of abelian groups on a topological space X. Then the *n-th cohomology of X with values in $\mathcal{E}$* is the abelian group $H^n(X, \mathcal{E}) := R^n\Gamma_X(\mathcal{E})$, i. e. the value of the n-th right derived functor of the global sections functor $\Gamma_X : \mathrm{AbSh}(X) \to \mathrm{Ab}$.[1]

Unrolling the definitions, the sheaf cohomology $H^n(X, \mathcal{E})$ is the cohomology of the cochain complex $\Gamma_X(\mathcal{I}^\bullet)$, where $0 \to \mathcal{E} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ is a resolution of $\mathcal{E}$ by injective objects. However, this description is rarely useful in practice. We will discuss techniques for calculating sheaf cohomology below.

Example 5.2. $H^0(X, \mathcal{E}) \cong \mathcal{E}(X)$.

Example 5.3. Sheaf cohomology on the one-point space $\mathrm{pt} = \{\heartsuit\}$ is trivial, in the following sense: $H^0(\mathrm{pt}, \mathcal{E}) \cong \mathcal{E}(\mathrm{pt}) \cong \mathcal{E}_\heartsuit$ and $H^{\geq 1}(\mathrm{pt}, \mathcal{E}) = 0$ for any sheaf $\mathcal{E}$. This is consistent with the well-known vanishing of ordinary (singular or simplicial) cohomology on the one-point space. An easy way to verify this is to note that $\Gamma_{\mathrm{pt}} : \mathrm{AbSh}(\mathrm{pt}) \to \mathrm{Ab}$ is exact (in fact, an equivalence of abelian categories). This implies that the higher derived functors vanish.

With singular or simplicial cohomology, the choice of a coefficient ring does not matter much in many applications; there is always the universal coefficient theorem relating $\mathbb{Z}$-valued cohomology with cohomology with values in other rings. *This is decidely not so with sheaf cohomology.* Sheaf cohomology is as much about the topological space as it is about the sheaf in question. The sheaf cohomology groups depend hugely on the used sheaf.

Unfortunately, there is no useful intuition about sheaf cohomology along the ways of "closed cochains modulo exact cochains". Instead, some intuition on what the elements of the sheaf cohomology groups look like can be gained by Čech methods (see below).

The basic proposition about sheaf cohomology is that short exact sequences of sheaves induce long exact sequences in cohomology.

[1] A sequence of sheaves is exact if and only if the induced sequences of stalks are exact, for all points of the space. It is a basic exercise in sheaf theory to verify that Γ_X is always left-exact. The standard example of an exact sequence which is not exact on global sections is the *exponential sequence* on $\mathbb{C} \setminus \{0\}$, $0 \to \underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^\times \to 0$. Here $\mathcal{O}$ denotes the sheaf of holomorphic functions and the morphism $\mathcal{O} \to \mathcal{O}^\times$ maps a holomorphic function f to $\exp \circ f$. This morphism is an epimorphism, since any nowhere vanishing holomorphic function h can *locally* be written as $\exp \circ (\log \circ f)$, where log is some locally-existing branch of the complex logarithm. But the induced map on global sections is not surjective. For instance, the function $z \mapsto 1/z$ has no preimage. To be a bit more dramatic: If the global sections functor were exact, then the subject "sheaf cohomology" would not exist.

Proposition 5.4. *Let X be a topological space. Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be a short exact sequence of sheaves of abelian groups on X. (This means that the induced sequences $0 \to \mathcal{E}_x \to \mathcal{F}_x \to \mathcal{G}_x \to 0$ on stalks are exact for all points $x \in X$.) Then there is a natural long exact sequence*

$$0 \to \mathcal{E}(X) \to \mathcal{F}(X) \to \mathcal{G}(X) \to H^1(X, \mathcal{E}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^2(X, \mathcal{E}) \to \cdots .$$

5.2. **Relation to ordinary cohomology.**

Proposition 5.5. *Let X be a locally contractible space. Let R be a ring and $\underline{R}$ be the constant sheaf with stalks R. Then $H^\bullet_{sing}(X, R) \cong H^\bullet(X, \underline{R})$.*

Proof. See [2]. The idea is to show that the sheafifications of the presheaves of singular chains, $U \mapsto C^n_{\text{sing}}(U, R)$, form a flabby resolution of the constant sheaf $\underline{R}$. To show that these sheaves form a resolution at all, one has to check that for a basis of open subsets U, the sequence $C^{p-1}_{\text{sing}}(U, R) \to C^p_{\text{sing}}(U, R) \to C^{p+1}_{\text{sing}}(U, R)$ is exact. By local contractibility, it suffices to consider contractible sets U, for which the exactness is well-known. □

XXX: Include comparison theorem with cohomology with values in local systems.

5.3. **Why sheaf cohomology?** Sheaf cohomology can be motived from many different angles.

The Leray spectral sequence. If $f : X \to Y$ is any continuous map, the cohomology of X, Y, and the fibers of f are related. The Leray spectral sequence expresses this relationship in a precise way, describing how the cohomology of X is composed by the cohomology of Y and the cohomology of the fibers. However, even if one is only interested in the cohomology of X with values in $\mathbb{Z}$ or in a field, the cohomology of non-constant sheaves appears in the spectral sequence. Only in special cases, such as that f is a fibration, cohomology with values in local systems suffices.

Global sections. In complex and algebraic geometry, one is often interested in the space of global sections of a sheaf $\mathcal{E}$. For instance, in the case that $\mathcal{E} = \mathcal{O}_X(D)$ is the line bundle associated to a divisor D on a complex manifold or scheme X, $H^0(X, \mathcal{E})$ is the space of meromorphic functions on X with zeroes and poles behaviour governed by D. However, this space, or even its dimension, is hard to compute. Therefore one turns to the *Euler characteristic* of $\mathcal{E}$, the alternating sum

$$\chi(X, \mathcal{E}) := \dim H^0(X, \mathcal{E}) - \dim H^1(X, \mathcal{E}) + \dim H^2(X, \mathcal{E}) \pm \cdots$$

in which the dimension of the space of global sections as the first summand. This quantity is in general easier to compute, since it is additive in short exact sequences (if $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is exact, then $\chi(X, \mathcal{F}) = \chi(X, \mathcal{E}) + \chi(X, \mathcal{G})$) and constant in certain families. Obviously, sheaf cohomology is needed to define the Euler characteristic.

Classifying geometric objects. For certain choices of the sheaf, sheaf cohomology classifies geometric objects. For instance, if X is a complex manifold, $H^1(X, \mathcal{O}_X^\times)$ classifies complex line bundles on X up to isomorphism. Here, $\mathcal{O}_X^\times$ is the sheaf of nowhere vanishing holomorphic functions. More generally, $H^1(X, \mathcal{A}ut(\mathcal{E}))$ classifies sheaves which are locally isomorphic to a fixed sheaf $\mathcal{E}$ and $H^1(X, \mathcal{G})$ classifies $\mathcal{G}$-torsors on X.

Irreducible spaces. The singular cohomology of irreducible topological spaces – spaces which cannot be written as the nontrivial union of two proper closed subsets – is always zero (in positive degrees). This is not a problem when working with manifolds, which are irreducible only in the one-point case. But it is a problem in algebraic geometry, where schemes with their Zariski topology are often irreducible. For such cases, one is forced to look at cohomology with values in non-constant sheaves.

5.4. **Higher direct images.**

Definition 5.6. Let $f : X \to Y$ be a continuous map. Let $\mathcal{E}$ be a sheaf of abelian groups on X. Then the *n-th higher direct image of $\mathcal{E}$ under f* is the sheaf $R^n f_*(\mathcal{E})$ of abelian groups on Y.

In this definition, $f_* : \mathrm{AbSh}(X) \to \mathrm{AbSh}(Y)$ denotes the left-exact pushforward functor and $R^n f_* : \mathrm{AbSh}(X) \to \mathrm{AbSh}(Y)$ denotes its n-th right derived functor.

Example 5.7. $R^0 f_*(\mathcal{E}) \cong f_*(\mathcal{E})$.

Example 5.8. Sheaf cohomology is a special case of higher direct images: Let $f : X \to \mathrm{pt}$ be the unique map to the one-point space. Then $H^n(X, \mathcal{E}) \cong R^n f_*(\mathcal{E})$ for any sheaf $\mathcal{E}$ of abelian groups on X.

For general maps $f : X \to Y$, the higher direct image can be pictured as a kind of "relativized" sheaf cohomology. To be more specific, we have the following results.

Proposition 5.9. *Let $f : X \to Y$ be a continuous map. Let $\mathcal{E}$ be a sheaf of abelian groups on X. Then $R^n f_*(\mathcal{E})$ is the sheafification of the presheaf $U \mapsto H^n(f^{-1}[U], \mathcal{E})$ on X.*

Proof. See [6, Tag 01E4]. □

Theorem 5.10 (Proper base change theorem)**.** *Let X be a topological space such that every open subspace is paracompact (for instance a manifold). Let Y be a Hausdorff space. Let $f : X \to Y$ be a closed continuous map. Let $\mathcal{E}$ be a sheaf of abelian groups on X. Then $(R^n f_*(\mathcal{E}))_y \cong H^n(f^{-1}[y], \mathcal{E}|_{f^{-1}[y]})$ for any point $y \in Y$.*

Proof. See [7, Thm. 8.11]. □

In the statement, the restriction $\mathcal{E}|_{f^{-1}[y]}$ is defined as the pullback of $\mathcal{E}$ under the inclusion map $i' : f^{-1}[y] \to X$ of the fiber in the total space.

The proper base change theorem is very useful for visualizing the higher direct image sheaves. The labeling "base change" is because the theorem can also be formulated in the following way: Consider the pullback (fiber product) diagram

$$\begin{tikzcd} f^{-1}[y] \arrow[r, hook, "i'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ \mathrm{pt} \arrow[r, hook, "i"'] & Y. \end{tikzcd}$$

Then $i^* R^n f_*(\mathcal{E}) \cong R^n f'_*(i'^* \mathcal{E})$, since pulling back along i is the same as calculating the stalk at y (example **XXX**).

There is also a more general base change theorem, valid for maps $i : Y' \to Y$ which are not necessarily the inclusion of a point.

Example 5.11. Let $\mathrm{id} : X \to X$ denote the identity map. Since its fibers are singletons, there should be no "relative cohomology" in positive degrees. Indeed, $R^n\mathrm{id}_*(\mathcal{E})$ is zero for any sheaf $\mathcal{E}$ and any $n > 0$. This can be verified directly with the definition (note that $\mathrm{id}_* : \mathrm{AbSh}(X) \to \mathrm{AbSh}(X)$ is the identity functor, which is exact). In case that the appropriate topological hypotheses are satisfied, it also follows from the proper base change theorem.

6. Čech methods

Definition 6.1. Let $\mathcal{E}$ be a sheaf (or presheaf) of abelian groups on a topological space X. Let $\mathcal{U} = (U_i)_i$ be an open covering. Then *the n-th Čech cohomology of $\mathcal{E}$ with respect to the covering $\mathcal{U}$*, $\check{H}^n(\mathcal{U}, \mathcal{E})$, is the n-th cohomology group of the *Čech cochain complex $\check{C}^\bullet(\mathcal{U}, \mathcal{E})$*, which is defined as follows:

$$\check{C}^n(\mathcal{U}, \mathcal{E}) := \prod_{i_0,\ldots,i_n} \mathcal{E}(U_{i_0\cdots i_n}),$$

where we set $U_{i_0\cdots i_n} := U_{i_0} \cap \cdots \cap U_{i_n}$ and use the differentials given by

$$d((s_{i_0\cdots i_n})_{i_0\cdots i_n}) := \left(\sum_{k=0}^{n+1} (-1)^k \, s_{i_0\cdots\widehat{i_k}\cdots i_{n+1}}|_{U_{i_0\cdots i_{n+1}}}\right)_{i_0\cdots i_{n+1}}.$$

Example 6.2. Let $\mathcal{E}$ is a sheaf and $\mathcal{U}$ be an arbitrary open covering of X. Then $\check{H}^0(\mathcal{U}, \mathcal{E}) \cong \mathcal{E}(X)$.

Note that some people argue [3] that Čech cohomology should not be termed "cohomology", since it fails to satisfy the demands one has for true cohomology theories (for instance, that short exact sequences induce long exact sequences in cohomology). Rather, it should be regarded as a tool for computing sheaf cohomology – we will discuss their relation in due course.

Unlike with sheaf cohomology, calculating Čech cohomology is in principle a viable linear algebra task. We can even restrict to the *ordered Čech complex*, given by

$$\check{C}^n_{\mathrm{ord}} = \prod_{i_0<\cdots<i_n} \mathcal{E}(U_{i_0\cdots i_n}),$$

where I is endowed with some fixed total ordering and the differential is given by the same formula. By an elementary result, this complex is homotopy equivalent to the original one [6, Tag 01FG] and has therefore isomorphic cohomology groups.

6.1. **The relation to sheaf cohomology.**

Čech cohomology and sheaf cohomology are related by a spectral sequence, the *Čech-to-cohomology spectral sequence*.

Proposition 6.3. *Let X be a topological space. Let $\mathcal{U} = (U_i)_i$ be an open covering of X. Let $\mathcal{E}$ be a sheaf of abelian groups on X. Then there is a spectral sequence with*

$$E_2^{pq} = \check{H}^p(\mathcal{U}, (U \mapsto H^q(U, \mathcal{E}))) \Longrightarrow H^n(X, \mathcal{E}).$$

Proof. See [6, Tag 01ES]. □

Remark 6.4. The presheaf of which the Čech cohomology is taken in the Čech-to-cohomology spectral sequence, $U \mapsto H^q(U, \mathcal{E})$, is in general not a sheaf. In fact, its sheafification is always zero [6, Tag 03BA].

In good cases, Čech cohomology does even agree with sheaf cohomology. This is the content of *Leray's theorem*.

Proposition 6.5 (Leray's theorem)**.** *Let X be a topological space. Let $\mathcal{U} = (U_i)_i$ be an open covering of X. Let $\mathcal{E}$ be a sheaf of abelian groups on X. Assume that $\mathcal{E}$ is acyclic on the intersections $U_{i_0 \cdots i_p}$, that is $H^q(U_{i_0 \cdots i_p}, \mathcal{E}) = 0$ for all $p \geq 0$ and $q > 0$. Then $H^n(X, \mathcal{E}) \cong \check{H}^n(\mathcal{U}, \mathcal{E})$ for all $n \geq 0$.*

Proof. The E_2 page of the Čech-to-cohomology spectral sequence is concentrated in the row $q = 0$, since E_2^{pq} is a subquotient of $\check{C}^p(\mathcal{U}, H^q(\_, \mathcal{E}))$, which is zero for $q > 0$. Therefore the spectral sequence degenerates. This implies the statement. $\square$

The vanishing assumption of the theorem is usually not checked by hand, but a result of *vanishing theorems*. For instance, each of the following conditions guarantees that $H^n(X, \mathcal{E})$ vanishes for all $n > 0$:

- The space X is contractible and locally contractible, and $\mathcal{E}$ is constant. (In this case, $H^n(X, \mathcal{E})$ coincides with singular cohomology, for which the vanishing is well-known.)
- The space X is an affine scheme and $\mathcal{E}$ is a quasicoherent $\mathcal{O}_X$-module. (This is *Serre's vanishing theorem*.)
- The space X is paracompact and the sheaf $\mathcal{E}$ is flabby, fine, or soft. Any sheaf of modules over the sheaf of continuous or smooth functions on a manifold is fine.

6.2. **The Mayer–Vietoris sequence from a higher point of view.**

Proposition 6.6. *Let $X = A \cup B$ be a covering of a topological space X by two open subsets. Let $\mathcal{E}$ be a sheaf of abelian groups on X. Then there is a Mayer–Vietoris sequence*

$$\cdots \longrightarrow H^n(X, \mathcal{E}) \longrightarrow H^n(A, \mathcal{E}) \oplus H^n(B, \mathcal{E}) \longrightarrow H^n(A \cap B, \mathcal{E}) \xrightarrow{\partial} H^{n+1}(X, \mathcal{E}) \longrightarrow \cdots.$$

Proof. Consider the Čech-to-cohomology spectral sequence for the given open covering $\mathcal{U}$. Since $\mathcal{U}$ consists only of two open sets, its E_2 page looks like this:

- E_2^{pq} is zero for $p \geq 2$.
- E_2^{0q} equals the kernel of the map $H^q(A, \mathcal{E}) \oplus H^q(B, \mathcal{E}) \to H^q(A \cap B, \mathcal{E})$ which sends (s, t) to $t|_{A \cap B} - s|_{A \cap B}$. (Use the alternating Čech complex.)
- E_2^{1q} is the cokernel of that map.

Therefore the spectral sequence degenerates on the E_2 page. Now consider, for any $n \geq 0$, the canonical short exact sequence

$$0 \longrightarrow F^1 E_\infty^n \longrightarrow E_\infty^n \longrightarrow E_\infty^n / F^1 E_\infty^n \longrightarrow 0$$

given by the filtration on E_∞^n. Since $F^2 E_\infty^n = F^3 E_\infty^n = \cdots = 0$ and $F^0 E_\infty^n = F^{-1} E_\infty^n = \cdots = E_\infty^n$ (by the vanishing of most columns, see for instance [4]), we can express the outer terms of this sequence in E_2 terms:

$$0 \longrightarrow E_2^{1,n-1} \longrightarrow E_\infty^n \longrightarrow E_2^{0,n} \longrightarrow 0.$$

Explicitly, this is the sequence

$$\begin{aligned} 0 &\longrightarrow \operatorname{cok}(H^{n-1}(A) \oplus H^{n-1}(B) \to H^{n-1}(A \cap B)) \\ &\longrightarrow H^n(X) \\ &\longrightarrow \ker(H^n(A) \oplus H^n(B) \to H^n(A \cap B)) \\ &\longrightarrow 0. \end{aligned}$$

We can splice these short exact sequences to obtain the long exact Mayer–Vietoris sequence. □

There is no Mayer–Vietoris like sequence for coverings consisting of more than two subsets; but the Čech-to-cohomology spectral sequence is still available in this case. Its E_2 page will have more nontrivial columns, and those columns will have greater combinatorial complexity.

6.3. **The colimit over all coverings.** To get rid of the dependence of Čech cohomology on the covering, one takes the colimit over all possible coverings: $\check{H}^n(X, \mathcal{E}) := \operatorname{colim}_{\mathcal{U}} \check{H}^n(\mathcal{U}, \mathcal{E})$.

Proposition 6.7. *Let $\mathcal{E}$ be a sheaf of abelian groups on a topological space X.*

(1) *Different refinements $\mathcal{U} \to \mathcal{V}$ induce homotopic maps $\check{C}^n(\mathcal{U}, \mathcal{E}) \to \check{C}^n(\mathcal{V}, \mathcal{E})$.*
(2) *There is a canonical map $\check{H}^n(X, \mathcal{E}) \cong H^n(X, \mathcal{E})$ which is an isomorphism for $n = 0$ and $n = 1$ and a monomorphism for $n = 2$.*
(3) *If X is paracompact, the map is an isomorphism for all n.*

Proof. See [1]. Also see the thread [5] on MathOverflow. □

- Čech intuition for sheaf cohomology.
- Special case: covering by a single open subset?

7. The Leray spectral sequence

Proposition 7.1. *Let $f : X \to Y$ be a continuous map. Let $\mathcal{E}$ be a sheaf of abelian groups on X. Then there is the* Leray spectral sequence

$$E_2^{pq} = H^p(Y, R^q f_*(\mathcal{E})) \Longrightarrow H^n(X, \mathcal{E}).$$

Proof. The mother of all spectral sequences is the spectral sequence associated to a filtered complex. Her first child is the Grothendieck spectral sequence; the Leray spectral sequence is a special case of it. Namely, the composition

$$\mathrm{AbSh}(X) \xrightarrow{f_*} \mathrm{AbSh}(Y) \xrightarrow{\Gamma_Y} \mathrm{Ab}$$

is the global sections functor Γ_X. Ignoring technicalities about adapted classes, we obtain the Grothendieck spectral sequence

$$E_2^{pq} = R^p\Gamma_Y(R^q f_*(\mathcal{E})) \Longrightarrow R^n\Gamma_X(\mathcal{E}).$$

This is the Leray spectral sequence. See [6, Tag 0732] for technical details. □

Example 7.2. In the situation of the proposition, assume that the higher direct images of $\mathcal{E}$ vanish (i. e. $R^q f_*(\mathcal{E}) = 0$ for $q > 0$). This is, for instance, satisfied if $\mathcal{E}$ has no higher cohomology on the fibers of f (and the assumptions of the proper base change theorem (Thm. 5.10) are fulfilled). Then the Leray spectral sequence degenerates on the second page and shows that $H^n(X, \mathcal{E}) \cong H^n(Y, f_*\mathcal{E})$.

Locally constant sheaves

Example 7.3. Consider S^1 together with the locally constant sheaf of typical stalk $\mathbb{Z}^2$ and monodromy representation $[\gamma].(a, b) = (b, a)$ where $[\gamma]$ is a generator of $\pi_1 S^1$. We want to calculate its sheaf cohomology via cohomology with local coefficient system. The bundle of coefficients E is given as

$$E := [0, 1] \times \mathbb{Z}^2 / \sim$$

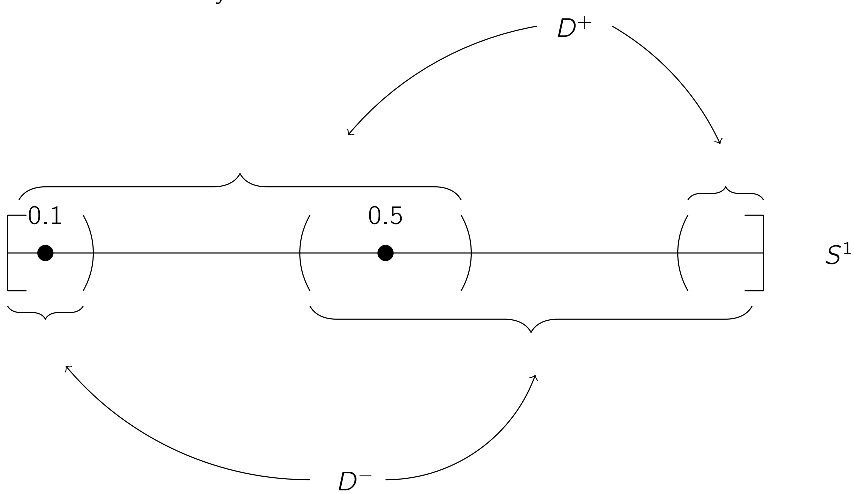
with the identification $(0, (a, b)) \sim (1, [\gamma].(a, b)) = (1, (b, a))$ and the obvious projection $E \to S^1$. We want to apply the Mayer-Vietoris sequence to the standard decomposition $S^1 = D^+ \cup D^-$ of S^1 into two open intervals yielding the exactness of

$$H^{k-1}(D^+, E) \oplus H^{k-1}(D^-, E) \longrightarrow H^{k-1}(D^+ \cap D^-, E) \longrightarrow H^k(S^1, E) \longrightarrow H^k(D^+, E) \oplus H^k(D^-, E)$$

On a simply connected space a local system of coefficients is globally constant and this shows that $H^k(S^1, E) = 0$ for $k > 1$. For $k = 1$ the sequence becomes

$$\begin{tikzcd} H^0(D^+, E) \oplus H^0(D^-, E) \arrow[r] \arrow[d] & H^0(D^+ \cap D^-, E) \arrow[r] \arrow[d] & H^1(S^1, E) \arrow[r] & 0 \\ \mathbb{Z}^2 \oplus \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2 \oplus \mathbb{Z}^2 & & \end{tikzcd} \tag{1}$$

We have to identify the lower arrow.



For an arbitrary space X a cochain representing a class in $H^0(X, E)$ assigns to every point $x \in X$ a value in the corresponding fibre E_x. The cochain condition means that this value is constant on connected components of X. There are no nontrivial coboundaries in this dimension. Therefore we have a very clear understanding of how the groups in the diagram 1 are isomorphic to $\mathbb{Z}^k$ namely by evaluation at some fixed points:

$$\begin{tikzcd} H^0(D^+, E) \oplus H^0(D^-, E) \arrow[r] \arrow[d, "\mathrm{ev}_{0.5}"', "\mathrm{ev}_{0.5}"] & H^0(D^+ \cap D^-, E) \arrow[r] \arrow[d, "\mathrm{ev}_{0.1} \oplus \mathrm{ev}_{0.5}"] & H^1(S^1, E) \arrow[r] & 0 \\ \mathbb{Z}^2 \oplus \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2 \oplus \mathbb{Z}^2 & & \end{tikzcd} \tag{2}$$

Consider the cohomology class in $H^0(D^+, E)$ represented by the cocycle which maps $0.5 \mapsto (a, b) \in \mathbb{Z}^2$. This cocycle restricted to D^+ also maps 0.1 to (a, b) because both of these points lie in the same connected component of D^+. Thus under isomorphisms 2 the restriction homomorphism $H^0(D^+, E) \to H^0(D^+ \cap D^-, E)$ maps $(a, b) \mapsto ((a, b), (a, b))$. On the other hand the cohomology class in $H^0(D^-, E)$ represented by the cocycle which maps $0.5 \mapsto (c, d) \in \mathbb{Z}^2$ maps $0.1 \mapsto (d, c)$ since we have to let the monodromy act first. Therefore the restriction homomorphism $H^0(D^-, E) \to H^0(D^+ \cap D^-, E)$ maps $(c, d) \mapsto ((d, c), (c, d))$. Mere calculation shows that the cokernel of $H^0(D^+, E) \oplus H^0(D^-, E) \to H^0(D^+ \cap D^-, E)$ is isomorphic to $\mathbb{Z}$ yielding

$$H^1(S^1, E) \cong \mathbb{Z}.$$

Direct Images

Example 7.4. Consider the 2-sheeted covering $p : S^1 \to S^1, z \to z^2$ viewing $S^1 \subset \mathbb{C}$. We want to describe the direct image $p_*\underline{\mathbb{Z}}$ of the constant sheaf $\underline{\mathbb{Z}}$. The sheaf $p_*\underline{\mathbb{Z}}$ is locally constant hence it is determined by its monodromy representation $\pi_1 S^1 = \mathbb{Z} \curvearrowright \mathbb{Z}^2$. The curve $\gamma : [0, 2\pi] \to S^1, t \mapsto \exp(it)$ is a generator of $\pi_1 S^1$ and we want to find a lift of this curve $I \to p_*\underline{\mathbb{Z}}$ to the étalé space of $p_*\underline{Z}$ with any given initial condition $\tilde{\gamma}(0) \in p_*\underline{\mathbb{Z}}_0$:

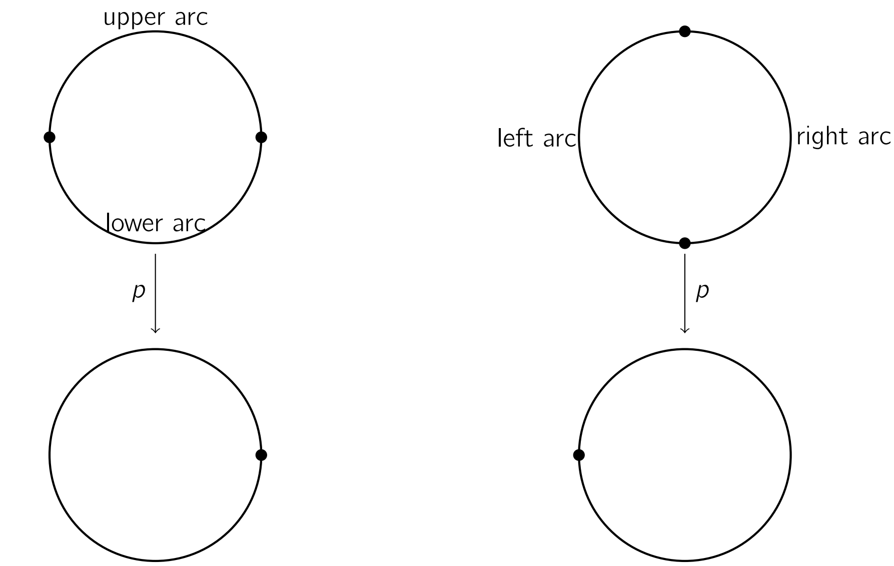
$$\begin{tikzcd} & p_*\underline{\mathbb{Z}} \arrow[d, "\pi"] \\ I \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & S^1 \end{tikzcd}$$

We choose two trivialisations

$$p_*\underline{\mathbb{Z}}S^1 \setminus \{+1\} = \underline{\mathbb{Z}}p^{-1}S^1 \setminus \{+1\} = \underline{\mathbb{Z}}S^1 \setminus \{\pm 1\} \xrightarrow{\phi} \mathbb{Z}_{upper} \oplus \mathbb{Z}_{lower}$$

$$p_*\underline{\mathbb{Z}}S^1 \setminus \{-1\} = \underline{\mathbb{Z}}p^{-1}S^1 \setminus \{-1\} = \underline{\mathbb{Z}}S^1 \setminus \{\pm i\} \xrightarrow{\psi} \mathbb{Z}_{left} \oplus \mathbb{Z}_{right}$$

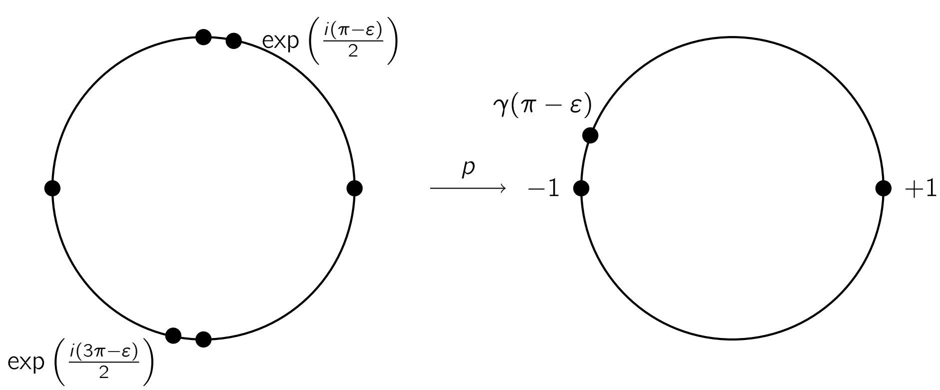
where for instance one generator of $\mathbb{Z}_{upper}$ corresponds to the locally constant real-valued function taking the value $+1$ on the upper connected arc of $S^1 \setminus \{\pm 1\}$ and vanishing on the lower one.



Let us describe the lift $\tilde{\gamma}$ with the initial condition $\tilde{\gamma}(0) = [\psi^{-1}(1,0)]_1$. At time $t = \pi - \varepsilon$ this lift still has the value $[\psi^{-1}(1,0)]_{\gamma(\pi-\varepsilon)}$ and we claim:

$$\tilde{\gamma}(\pi - \varepsilon) = [\psi^{-1}(1,0)]_{\gamma(\pi-\varepsilon)} \tag{3}$$
$$= [\phi^{-1}(0,1)]_{\gamma(\pi-\varepsilon)} \tag{4}$$

The point $\tilde{\gamma}(\pi - \varepsilon)$ is obtained as follows: consider the locally constant real-valued function $f \in p_*\underline{\mathbb{Z}}(S^1 \setminus \{-1\}) = \underline{\mathbb{Z}}(S^1 \setminus \{\pm i\})$ having the value 1 on the left arc and 0 on the right one. This is a section of $p_*\underline{\mathbb{Z}}$ over the open set $S^1 \setminus \{-1\}$. The preimage of $\gamma(\pi - \varepsilon)$ consists of the two points $\exp\left(\frac{i(\pi-\varepsilon)}{2}\right)$ and $\exp\left(\frac{i(3\pi-\varepsilon)}{2}\right)$:



On the other hand the real-valued function $g := \phi^{-1}(0,1) \in p_*\underline{\mathbb{Z}}(S^1 \setminus \{+1\}) = \underline{\mathbb{Z}}(S^1 \setminus \{\pm 1\})$ having the value 0 on the upper arc and 1 on the lower one represents exactly the same stalk of $p_*\underline{\mathbb{Z}}$ at the point $\gamma(\pi - \varepsilon)$. This proves claim (4).

A similar argument shows

$$\begin{aligned}\tilde{\gamma}(2\pi-\varepsilon) &= [\phi^{-1}(0,1)]_{\gamma(2\pi-\varepsilon)}\\ &= [\psi^{-1}(0,1)]_{\gamma(2\pi-\varepsilon)}\end{aligned}$$

and then $\tilde{\gamma}(2\pi) = [\psi^{-1}(1,0)]_1$. Together with a similar calculation on the other generator of the stalk at 1 we conclude that in the monodromy representation $\pi_1 S^1 \curvearrowright \mathbb{Z}^2$ the generator of $\pi_1 S^1$ interchanges the two summands of $\mathbb{Z}^2$ – the stalk of $p_*\underline{\mathbb{Z}}$ at the point $1 \in S^1$.

The Serre Spectral Sequence

Example 7.5. Every covering is a fibration and hence yields a Serre spectral sequence converging to the cohomology of the total space. Revisiting Example 7.4 we observe that every preimage of a point consists of two discrete points and hence alle higher direct images $R^i p_*\underline{\mathbb{Z}}, i > 0$ vanish. Therefore the Serre spetral sequence looks as follows:

$$\begin{array}{cccc}
1 & & & \\
0 & H^0(S^1, p_*\underline{\mathbb{Z}}) & H^1(S^1, p_*\underline{\mathbb{Z}}) & H^2(S^1, p_*\underline{\mathbb{Z}}) \\
 & 0 & 1 & 2
\end{array}$$

Thus the E_2 page equals the limiting page E_∞ and we have an isomorphism

$$H^k(S^1, p_*\underline{\mathbb{Z}}) \cong H^k(S^1, \mathbb{Z})$$

for any $k \geq 0$. The left hand side expression is sheaf cohomology and the right hand one may be interpreted as singular cohomology with coefficients in $\mathbb{Z}$.

For $k = 0$ this is obvious since $H^0(S^1, p_*\underline{\mathbb{Z}}) = \Gamma p_*\underline{\mathbb{Z}} = \mathbb{Z}$. For $k > 0$ this is some statement about the singular cohomology of S^1 with local coefficient system $p_*\underline{\mathbb{Z}}$ of typical fibre $\mathbb{Z}^2$ (cf. 7.3).

Example 7.6. Let $M = S^n$ and

$$S^{n-1} \hookrightarrow SM \to S^n$$

its unit sphere bundle. We want to calculate the integral homology of SM. The E^2 entries are

$$\begin{array}{ccc}
n & \mathbb{Z} & \mathbb{Z} \\
0 & \mathbb{Z} & \mathbb{Z} \\
 & 0 & n-1
\end{array}$$

- definition
- many examples, also with differentials and ring structure

- Klein bottle
- Unit sphere tangent bundle of spheres

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